A Note on Dawson's Chess

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Among many imaginative fairy chess problems of T. R. Dawson (1935), Problem 80 asks for the solution of a game that has become to be known as Dawson's Chess.

> *Given two equal lines of opposing Pawns, White on 3rd rank, Black on 5th, n adjacent files, White to play, at losing game, what is the result?*

It is understood that a capture must be made when possible.

In terms of removing counters from piles, the rules may be described as follows: (1) a pile consisting of a single counter may be removed, (2) two counters may be removed from any pile, and (3) three counters may be removed from any pile and if desired that pile may be split into two parts. This game is a member of the class of octal games of Guy and Smith (1956) — specifically .137 in their notation. Under the rule that the last player to move wins, Guy and Smith show that the game has a remarkable analysis with a Grundy function eventually periodic of period 34.

However, under the rule proposed by Dawson that the last player to move loses, the game becomes more difficult to analyze. As a partial analysis, Dawson gave some tentative results.

> *For small values of n, up to at least 50, first player loses if n equals 1, 2, 6, 7, or 11, modulus 14. In the case of remainder 4, mod. 14, the first player wins whatever move he plays first, e.g. for cases 4, 18, and 32 files.*

Since a straightforward analysis listing winning positions becomes exceedingly tedious for values of n beyond 20, one wonders how Dawson, so obviously gifted with combinatorial skills, carried out his analysis to $n = 50$.

In 1974 with the aid of a computer at UCLA, it was discovered that Dawson must have made an error in his analysis, since (1) the first player can win with $n = 43$ by moving the central pawn (eliminating the three central files), and (2) for $n = 32$, the first player can make a losing move by moving the 5th or 11th pawn from one end.

In this note, we show these two facts without the aid of a computer, using the analysis of misére games developed by Conway (1976). We then give a complete analysis of Dawson's chess when all piles are of size less than 20.

For this purpose, we use Table 1, a table of simplified positions for Dawson's chess for pile sizes up to 20. A version of this table may be found in Berlekamp, Conway and Guy (1982) page 425, under the analysis of .07 which is a first cousin to .137, and in Conway (1976) page 145, under the analysis of .4 which is a second cousin to .137. Although the positions simplify greatly for sizes up to 19, there is much less simplification beyond that point. An extension of this table, giving the simplified positions for piles up to size 36 has been carried out by Jim Flanigan. Fortunately, it is sufficient to study piles of size 20 and less to be able to disprove Dawson's claim. This is because the (unique) winning move for

n:	0	1	2	3	4	5	6	7	8	9	
G_n:	0	1	1	2	0	3	1	1	0	a	
n:	10	11	12	13	14	15	16	17	18	19	20
G_n:	3	b	2	c	0	d	a_1	2	b_1	3	g

Table 1. List of Simplified Positions for Dawson's Game for n up to 20, where $a = 2_2 21$, $b = a2_3 0$, $c = ba_1 2_2 31$, $d = cb_1 a_2 20$, and $g = d_1 cb_1 a 2_3 2$.

+	a	b	c	d	g
0	3^{143}	2^{052}	4^{14}	5^{05}	0^{0}
a	0^{12}	1^{03}	7^{17}	6^{06}	3^{3}
b		0^{12}	6^{06}	7^{17}	2^{2}
c			0^{12}	1^{03}	4^{586}
d				0^{12}	5^{497}
g					0^{0}

Table 2. $\mathcal{G}^*$-values of sums of a, b, c, d, and g.

a pile of size 43 is to remove 3 and break it into two piles of size 20. Thus, all we have to do is to show that the position consisting of two piles of size 20, namely, $g + g$, is a P-position, i.e. a previous player win.

For this, it is sufficient to compute the genus, or $\mathcal{G}^*$-value, of g. In Table 2, we present the genus of the sum of any two relevant positions. This enables one to play the game correctly provided there are at most two piles of the type a, b, c, d, a_1, b_1, or g. In particular, we see that the genus of $g + g$ is 0^0, where the superscript zero indicates that $g+g$ is a (misére) P-position. Also, we see that a player faced with a single pile of size 32, can make a losing move by moving to two piles of sizes 9 and 20.

By a slightly deeper analysis, we can give a complete solution to the game with an arbitrary number of piles provided all piles are of size less than 20. Computation of the $\mathcal{G}^*$-values of sums of arbitrary numbers of a's, b's, c's and d's is sufficient for this purpose. It turns out that provided no pile is of size 20 or greater, b behaves like $a+1$, and d behaves like $c + 1$, so it is sufficient to list the $\mathcal{G}^*$-values of sums of arbitrary numbers of a's and c's. This is done in Table 3.

	N(a) even	N(a) odd
$N(c) = 0$	0^{12}	3^{143}
$N(c)$ odd	4^{14}	7^{17}
$N(c)$ even ≥ 2	0^{12}	3^{183}

Table 3. $\mathcal{G}^*$-values for sums of an arbitrary number of a's and c's.

We conclude by describing the P-positions for Dawson's Chess when all piles are of size 19 or less. Every position may be reduced to sums of positions of the types 0, 1, 2, a

and c, by replacing b by $a+1$, d by $c+1$, $a_1 = a+1$, $b_1 = b+1$, and $3 = 2+1$. Let $N(i)$ denote the number of positions of type i when reduced, $i = 0,\ 1,\ 2,\ a,\ c$. A position is a P-position if and only if

(1) $N(2) = 0$ and $N(1)$ is odd, or

(2) $N(2) \geq 2$ and $N(a) + N(2)$ is even and $N(a) + N(1)$ is even and $N(c)$ is even.

As an example consider Dawson's Chess with piles of sizes 7,9,11,13,15,16,17,18 and 19, which represent respectively positions of the type 1, a, $a+1$, c, $c+1$, $a+1$, 2, a, $2+1$. Thus, $N(1) = 5$, $N(2) = 2$, $N(a) = 4$, $N(c) = 2$. Since $N(2) = 2$ and $N(a) + N(1)$ is odd, this is not a P-position. It can be made into a P-position by dropping one of the 1's, or by adding another 1 (e.g. $7 \to 4$; $11 \to 9$; $15 \to 13$; $16 \to 4,9$; $19 \to 4,12$, $18 \to 16$, or $18 \to 4,11$).

References.

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