WORKING NOTES:
Games at Dal 4

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NOTES PREPARED BY: Angela Siegel

1 Topics to Consider

- simplifying misere (disjoint sums)
- impartial misere clobber
    - sequential joins in
- sequential joins and identity
- algorithms for finding uptimal games
    - $0.2\bar{1}4 \approx 2\cdot \uparrow + \downarrow_2 + 4\cdot \uparrow^3$
    - 0.214 (base G)
- continue 0.13 calculation
- subtraction
    - L can subtract 1 or 2, R can subtract 1 or 7 (or 1 or 3)
    - quotient
- cutthroat stars

Monday, 21 August 2006 - 9:30am

2 Thane Plambeck

References:

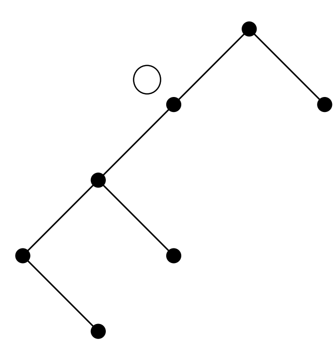
- arXiv: math.co/0603027
- arXiv: math.co/0501315
- "G-values of Various Games"
- "Φ-values of Various Games"

Impartial games:

- 2 players
- complete information
- no randomness
- from any position, options are the same for each player
- no loops

Game: Move coin "down" until we reach terminal vertex. In normal play, the person who slides last coin wins. In misere, he loses.

Example 1

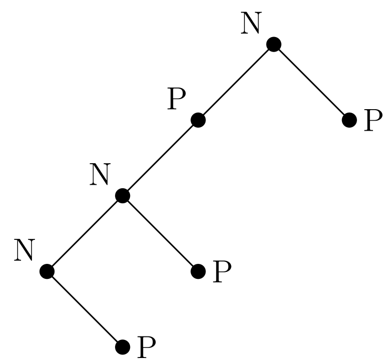


In normal play, want to go second. In misere, still want to go second.

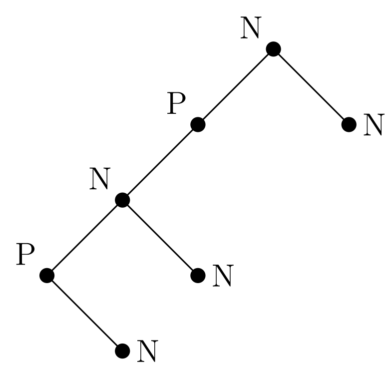
Every impartial game (normal or misere) is either an $\mathcal{N}$-position or a $\mathcal{P}$-position.

$\mathcal{N}$-position: Next player wins in best play.
$\mathcal{P}$-position: Previous player wins in best play.

Normal play (terminal positions are $\mathcal{P}$-positions)

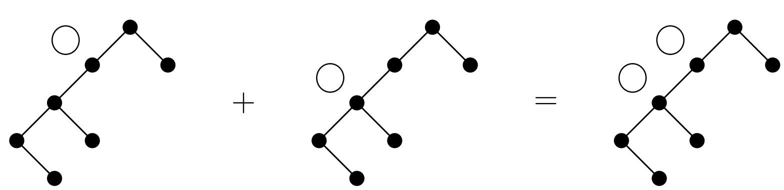


Misere play (terminal positions are $\mathcal{N}$-positions)



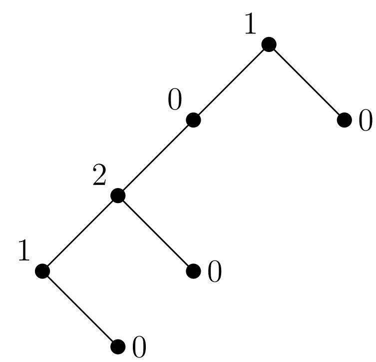
2.1 Sums

The following will denote disjunctive sums of games. Will draw as one tree with multiple coins.



2.2 Sprague-Grundy (S/G) Theory for Normal Play

- Put "0" at leaf nodes
- For other vertices v, label(v)=mex(options(v))
  where mex is the minimal excludant.



Nim addition $\oplus$

$n \oplus m$

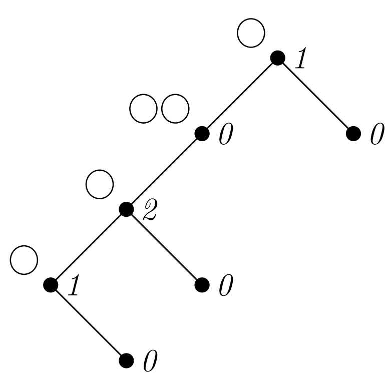
$= \text{Bit XOR } (n, m) =$ adding in base 2 without carrying.

Example 2

$$\begin{array}{rcl} 2 & = & (10)_2 \\ \oplus\, 3 & = & (11)_2 \\ \hline 1 & = & (01)_2 \end{array}$$

A position is a $\mathcal{P}$-position if it evaluates to zero.

Example 3



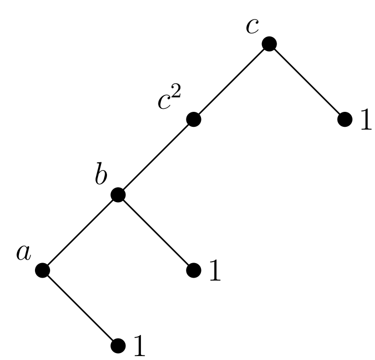
$1 \oplus 0 \oplus 0 \oplus 2 \oplus 1 = 2$.

$\therefore \mathcal{N}$*-position (normal play).*

2.3 Misere play

Monoid $Q =< a, b, c | a^2 = 1, b^3 = b, b^2c = c, c^3 = ac^2 >$

- · commutative monoid presentation
- · identity=1
- · written multiplicative
- · Set of all $\{a^i b^j c^k | i, j, k \geq 0\}$ subject to relations.
- · 14 elements: $\{1, a, b, c, ab, ac, b^2, bc, c^2, ab^2, ac^2, bc^2, abc, abc^2\}$
- · confluent presentation, Knuth-Bendix rewriting process



Claim: A position is a $\mathcal{P}$-position if and only if it reduces to a, b^2, bc or c^2.

Q: What is going on with this? Where did THIS come from?

2.4 Misere Indistinguishability Quotient $\equiv$ Misere Quotient

$\mathcal{A}$ = set of impartial games closed under addition and making moves.

G, H typical position in $\mathcal{A}$.

Indistinguishability relation ρ

$$G \rho H \qquad \equiv \qquad G + X \text{ is } \rho \iff H + X \text{ is } \rho \qquad \forall X \in \mathcal{A}.$$

- For Normal or Misere play (for normal play, recovers S/G value).
- ρ is a congruence on $\mathcal{A}$
- $\rho G = \{H | G \rho H\}$
- $\rho G + \rho H = \rho(G + H)$
- monoid $Q = Q(\Gamma) = \mathcal{A}/\rho$ (where Γ defines game)

2.5 Problems

1. How do we compute misere quotients?

   - Aaron Siegel wrote *MisereSolver* (java)
   - How to do this for infinite quotients?
     - Redei (1960's): A finitely generated commutative monoid is always finitely presented.
     - Finite # of generators, infinite # of rules, can present with finite # of rules. (Consequence of Hilbert Basis Thm)

2. How do we verify they are correct?

   - algorithm if finite

3. What is the algebraic structure (category) of these objects?

   - If you read off as group presentation, you get cross product of $\mathbb{Z}_2$'s. ($\mathbb{Z}_2^n$)

     e.g.

     $< a^2 = 1, b^3 = b, b^2c = c, c^3 = ac^2 >$

     $< a^2 = 1, b^2 = 1, c = a >$

     $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4. What is the relationship between normal and misere quotients?

   - Normal Kernel Hypothesis

5. What is the analog of the mex rule in misere play?

- Transition algebra (Aaron Siegel)

6. What about infinite ones?

7. What about other games?

- Dawson's Chess (1935)

In the 1930's Sprague-Grundy theory was developed in C.A.B Smith and Guy's "G-Values of Various Games". Showed that complicated structures = $*k$ for some k.

Smith and Grundy showed in "Disjoint Games with Last Player Losing" that complicated trees = complicated trees (with some simplification rules), but couldn't prove.

In 1970's, Conway showed that trees indeed can't be simplified further and that Smith/Grundy were correct/complete.

Tame game = game that can be treated in misere play as if it were Misere Nim.

Wild game = a game that is not tame.

How to calculate whether a game is tame? Genus symbols!

2.6 Nim vs. Misere Nim

Nim

Heaps of beans. Rule: Take as many (possibly all) beans from one of the heaps.

Example 4

5 6 4 1

$$\begin{array}{rcl} 5 & = & 101_2 \\ 6 & = & 110_2 \\ 4 & = & 100_2 \\ \oplus\, 1 & = & 001_2 \\ \hline 6 & = & 110_2 \end{array}$$

$\therefore \mathcal{N}$*-position in normal play.*

Misere Nim

Play just as in normal play unless your move leads to heaps of size 1 only.

Example 5

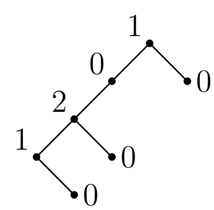
$1 + 3$

In normal play, would play to 1+1, but this would not work in misere. Must play to 1+0.

2.7 Sprague-Grundy Numbers

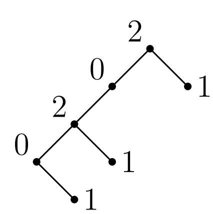
Normal Play S/G Number

$$G^+(G) = \begin{cases} 0 & \text{if } G \text{ has no options} \\ mex & \text{otherwise} \end{cases}$$



Misere Play S/G Number

$$G^-(G) = \begin{cases} 1 & \text{if } G \text{ has no options} \\ mex & \text{otherwise} \end{cases}$$



A game in misere is a $\mathcal{P}$-position $\iff$ the misere S/G value is 0.

In normal play, if we know $G^+(G)$ and $G^+(H)$, then $G^+(G+H)$ is known.

In misere play, if we know $G^-(G)$ and $G^-(H)$, then $G^-(G+H)$ could be anything!

2.8 Genus

$\text{Genus}(G) = G^+(G)^{G^-(G)\,G^-(G+\hat{2})\,G^-(G+\hat{2}+\hat{2})\,\ldots}$

$0 = \{\ \}$

$\text{Genus}(0) = 0^{1202020\ldots}$ denoted 0^{120}.

since

$G^+(0) = 0$
$G^-(0) = 1$
$G^-(0+\hat{2}) = 2$
$G^-(0+\hat{2}+\hat{2}) = 0$

If you know the genus of the options of G, you can "easily" calculate the genus of G.

Example 6

If we know Genus$(G_1) = a^{a_1\,a_2\,a_3}$ *and Genus*$(G_2) = b^{b_1\,b_2\,b_3}$*, for a game* $G = \{G_1, G_2\}$*, what is Genus(G)?*

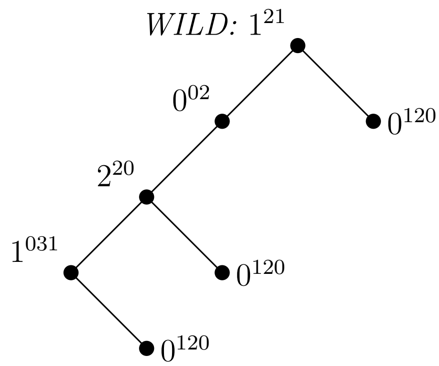
$$\begin{array}{c} a^{a_1\,a_2\,a_3\,\ldots} \\ b^{b_1\,b_2\,b_3\,\ldots} \\ \hline c^{c_1\,c_2\,c_3\,\ldots} \end{array}$$

where

$c = mex(a, b)$,
$c_1 = mex(a_1, b_1)$
and for $n > 1$, $c_n = mex(a_n, b_n, c_{n-1}, c_{n-1} \oplus 1)$.

So since we know Genus$(0) = 0^{120}$ *and Genus*$(1) = 1^{031}$*, we know Genus(2)=*2^{20}*:*

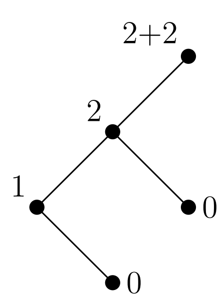
$$\begin{array}{c} 0^{120} \\ 1^{031} \\ \hline 2^{202\ldots} \end{array}$$



2.9 Tame Genera

$$\begin{array}{cl} 0^{120} & = \text{even \#'s} \\ 1^{031} & = \text{odd \#'s} \\ \hline 2^{20} & \\ 3^{31} & = \text{Genus(nim heap size 3)} \\ 4^{46} & \\ 5^{57} & \\ 6^{64} & \\ \vdots & \\ \hline 0^{02} & = {*0} \text{ in normal play but with at least one heap} > 1 \\ 1^{13} & = {*1} \text{ in normal play but with at least one heap} > 1 \end{array}$$

Can pretend nim heaps of size:



What happens if it's wild?

#coins at root	genus
0	0^{120}
1	1^{20}
2	0^{02}
3	1^{13}
4	0^{02}
$\vdots$	$\vdots$

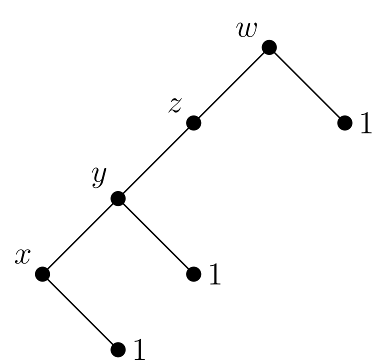
In normal play, $G + G = 0$ (and so therefore $G + G + G = G$), but this is rarely true in misere play.

However, in misere play, we will have
$G + G + G \;\; \rho \;\; G$
or $\quad G + G + G + G \;\; \rho \;\; G + G$
or $\quad G + G + G + G + G \;\; \rho \;\; G + G + G$
or for some k, $\quad \sum_{i=1}^{k+2} G \;\; \rho \;\; \sum_{i=1}^{k} G$

i.e. $< x_i | x_i^{k+2} = x_i^k \text{ for some } k >$.

2.10 LEAP OF FAITH!

We don't know what the values should be, so let's introduce the generators w, x, y and z as follows:



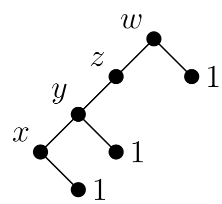
We are then interested in something like $< x, y, z, w \mid x^4 = x^2, y^3 = y, z^2 = 1, w^5 = w >$.

Order of this is 120:

- 4 choices for x: $\quad 1, x, x^2, x^3$
- 3 choices for y: $\quad 1, y, y^2$
- 2 choices for z: $\quad 1, z$

- 5 choices for w: $1, w, w^2, w^3, w^4$

Outcomes:



- 1: $\mathcal{N}$-position
- x: $\mathcal{P}$-position
- y: $\mathcal{N}$-position
- z: $\mathcal{P}$-position
- w: $\mathcal{N}$-position
- x^2 (2 coins on x): $\mathcal{N}$-position

To prove that this works, we must show:

(A) Everything I say is an $\mathcal{N}$-position has a move to a $\mathcal{P}$-position, provided not endgame. (exponential)

(B) If I say that it's a $\mathcal{P}$-position, then all options are to $\mathcal{N}$-positions. (polynomial)

We have a putative quotient Q and inferred $\mathcal{P}$ and $\mathcal{N}$ positions. How do we know it's works out alright for a complex position like $x^{10}y^{12}w^{13}z^{101}$?

(B) Show that if $\mathcal{P}$, all options are to $\mathcal{N}$. Assume $\mathcal{P} \to \mathcal{P}$. Must check for any position $x^i y^j z^k w^l$.

Possible moves	As ordered pairs (s, t)
$w \to 1$	$(w, 1)$
$w \to z$	(w, z)
$y \to 1$	$(y, 1)$
$y \to x$	(y, x)
$x \to 1$	$(x, 1)$

Interested in set $S = \{x^i y^j z^k w^l (s,t)\}$ [contains (xw, x), (xw, xz), etc.] in $Q \times Q$.

$Q \times Q$ has 120 elements. Show that $(\mathcal{P}, \mathcal{P})$ doesn't happen.

Suggestion (Selinger): This looks like co-algebra / co-induction.

Monday, 21 August 2006 - 1:00pm

3 Peter Selinger: Co-induction

3.1 Induction

Consider the largest subset A of $\mathbb{N}$ s.t.
(i) $2 \in A$
(ii) $x \in A \implies 3x \in A$
(iii) $x \in A, y \in A \implies 2x + y \in A$.

(a) Prove $10 \in A$.

$$2 \in A \quad \implies \quad 3(2) = 6 \in A \quad \implies \quad 2(2) + 6 = 10 \in A.$$

(b) Prove $5 \notin A$.

Must prove by induction that all elements of A are even.

(c) Prove $4 \notin A$.

By induction, all elements of A are 2 or ≥ 5.

Formally:

Let P be some property of numbers. To prove that $\forall x \;\; x \in A \to P(x)$, it suffices
(i) $P(2)$
(ii) $P(x) \to P(3x)$
(iii) $P(x), P(y) \to P(2x + y)$.

Proof: Let $B = \{x | P(x)\}$.

On what kind of structures is there an induction principle?

Let X be a set. Let $c : Pow(X) \to Pow(X)$ be some monotone operation on subsets of X. We say $B \subseteq X$ is closed under c if $c(B) \subseteq B$.

Claim:
(a) There exists a smallest set A closed under c.
(b) Induction principle to prove $A \subseteq B$ suffices $c(B) \subseteq B$.

To show (a), consider $\{B_i\}_{i \in I}$ the family of all closed sets. Let $A = \bigcap_{i \in I} B_i$.

$$c(A) = c(\bigcap_{i \in I} B_i) \subseteq c(B_i) \subseteq B_i$$

$$\implies c(A) \subseteq \bigcap_{i \in I} B_i = A.$$

Lemma: If A is smallest set such that $c(A) \subseteq A$, then $c(A) = A$.

To show $A \subseteq c(A)$, suffices to show $c(c(A)) \subseteq c(A)$.

Example 7

Let G be a monoid, $a, b \in G$. Let $\sim$ be the smallest congruence such that $a \sim b$.

$\sim \;\; \in \;\; G \times G$.

(1) $a \sim b$
(2) $x \sim x$
(3) $x \sim y \;\; \to \;\; y \sim x$
(4) $x \sim y, \;\; y \sim z \;\; \to \;\; x \sim z$
(5) $x \sim y \;\; \to \;\; xz \sim yz$.

$c(\sim) \subseteq \sim$

$c : Pow(G \times G) \to Pow(G \times G)$

$cR = \{(a, b)\} \cup \{(x, x) | \, x \in G\} \cup \{(y, x) | \, xRy\} \cup \{(x, z) | \, xRy, \;\; yRz\} \cup \ldots$.

3.2 Co-induction

Consider the smallest subset A of $\mathbb{N}$ s.t.
(i) $100 \notin A$
(ii) $x \in A \implies 2x \in A$.

(a) Prove $25 \notin A$.

Directly. Suppose $25 \in A \implies 2(35) = 50 \in A \implies 2(50) = 100 \in A$. This leads to a contradiction. So $25 \notin A$.

(b) Prove $24 \in A$.

By co-induction, we can prove that all numbers divisible by 3 are elements of A. Let B be the set of all number divisible by 3. Then (1) $100 \notin B$. Also (2), $x \in B \to 2x \in B$. Since A was the largest, $B \subseteq A$.

Induction: All elements in A have a certain property.

vs.

Co-induction: All elements with a certain property are in A.

(b) Prove $5 \in A$.

Let $B = \{x | \, 25 \nmid x\}$. $5 \in B \subseteq A$.

On what kind of structures is there an co-induction principle?

Let X be a set. Let $c : Pow(X) \to Pow(X)$ be some monotone operation on subsets of X. We say $B \subseteq X$ is co-closed under c if $c(B) \subseteq B$.

Claim:
(a) There exists a smallest set A co-closed under c.
(b) Co-induction principle to prove $A \supseteq B$ suffices $c(B) \supseteq B$.

To show (a), consider $\{B_i\}_{i \in I}$ the family of all co-closed sets. Let $A = \bigcup_{i \in I} B_i$.

$$c(A) = c(\bigcup_{i \in I} B_i) \supseteq c(B_i) \supseteq B_i$$

$$\implies c(A) \supseteq \bigcup_{i \in I} B_i = A.$$

Lemma: If A is smallest set such that $c(A) \supseteq A$, then $c(A) = A$.

Example 8

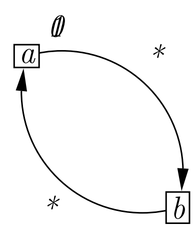
From previous example, $c(A) : \; x \in A \implies x \neq 100 \wedge 2x \in A$.
So, $A \subseteq c(A) = \{x | \, x \neq 100 \wedge 2x \in A\}$.

State machine. Let B be a finite set of buttons and L be a finite set of lights. A state machine is a 3-tuple $< S, next, value >$ where S is a set (called the set of states), $next : B \times S \to S$, $value : S \to L$.

Example 9

$B = \{*\}$

$L = \{0, 1\}$



We say $S_0 \sim S_1$ if

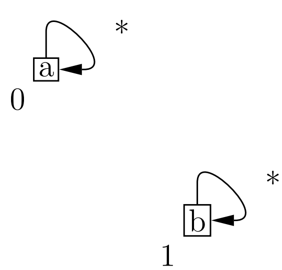
(1) $value(S_0) = value(S_1)$ (i.e. both have the same lights on/off) and

(2) $S_0 \sim S_1 \to \forall b \in B,\ next(b, S_0) \sim next(b, S_1)$.

Let $\sim$ be the largest relation R satisfying

(1) $S_0 R S_1 \to value(S_0) = value(S_1)$

(2) $S_0 R S_1 \to \forall b \in B,\ next(b, S_0) \sim next(b, S_1)$



How do we compute $\sim$? If the set of states is finite, there is an algorithm.

$c : Pow(X) \to Pow(X)$

$A_0 = X$

$A_{n+1} = c(A_n)$ (so $A_1 \subseteq A_0$, etc.)

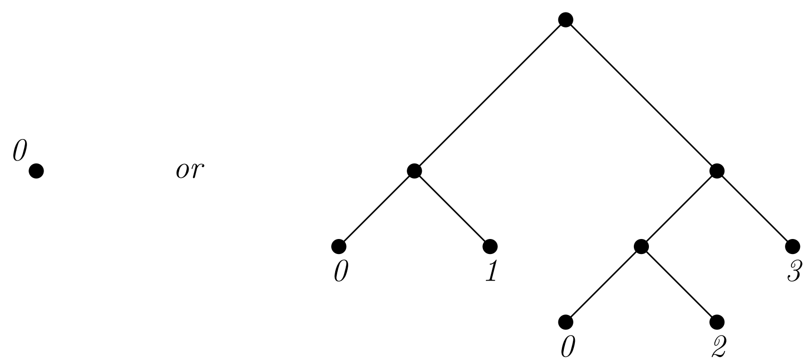
$A = \bigcap_n A_n$

What if the set of states is infinite?

Suppose that you already have R satisfying (1) and (2) and such that S/R is finite.

Example 10

(An inductive definition) Let L be a set of labels. A binary tree labeled by L is either (i) an element of L or (ii) a pair (t_1, t_2) of two L-labeled trees.



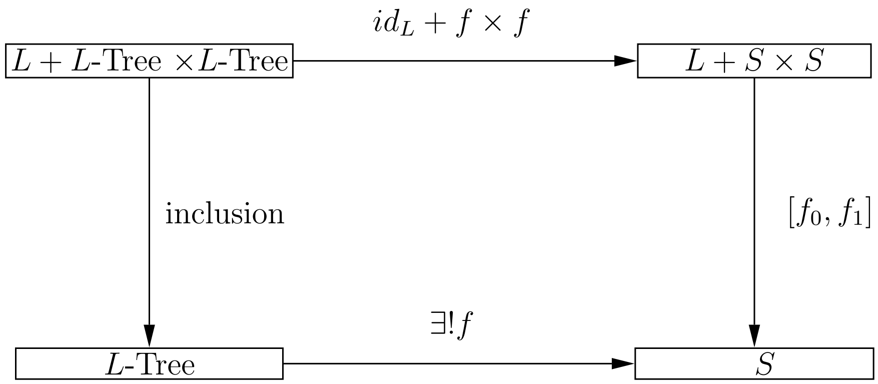
Formally:

The set L-Tree is the initial smallest set such that
(1) $L \subseteq L$-Tree and
(2) L-Tree $\times L$-Tree $\subseteq L$-Tree.

Recursion

To define a function $f : L\text{-Tree} \to S$, it suffices to define $f_o : L \to S$ and $f_1 : L \to S$ and then there will exist unique f such that
$f(l) = f_0(l)$ and
$f(t_1, t_2) = f_1(f(t_1), f(t_2))$

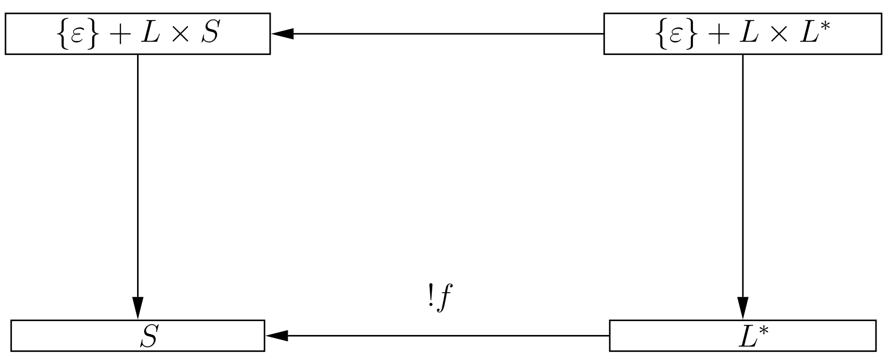
$L + L\text{-Tree} \times L\text{-Tree} \subseteq L\text{-Tree}$.



Example 11

Let L be a set (of generators). An L-word is either
(1) the empty word ε or
(2) a pair $l \cdot w$ of $l \in L, w \in L$-word

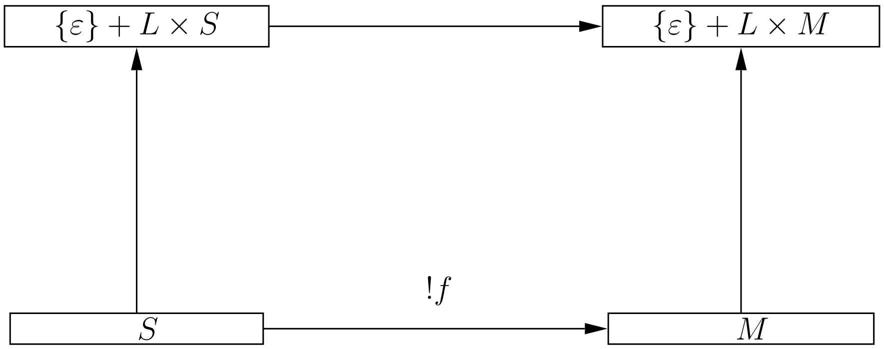
Equivalently, $L^* = \{\varepsilon\} + L \times L^*$.



Co:

$M = L^* \times L^\infty$.

Works on finite and infinite L-words.



Induction corresponds to data types or output.
Co-induction corresponds to systems or input.

MONDAY, 21 AUGUST 2006 - 2:30PM

4 Thane Plambeck

4.1 0.123

Rules:

- May take 1 bean if isolated
- May take 2 beans if at least 2 in heap
- May take 3 beans under any circumstances

Heap size 8:

$G(x)$	1	2	3	4	5
0+	1	0	2	2	1
5+	0	0	2	1	1
10+	0	0	2	1	1
15+	…				

Normal Play

Genera	1	2	3	4	5
0+	1^{031}	0^{120}	2^{20}	2^{20}	1^{031}
5+	0^{02}	0^{120}	$\mathbf{2^{1420}}$	1^{20}	1^{031}
10+	0^{02}	0^{120}	…		
15+	…				

Misere

Genera table is just used to locate where the game (at heap size 8) first went wild, i.e. at 2^{1420}.

4.2 Tame Quotients

First tame quotient:

$$T_1 =< a|\, a^2 = 1 >= \{1, a\}.$$

2 elements, game is equivalent to Nim with heaps of size 1.

Second tame quotient:

$$T_2 =< a, b|\, a^2 = 1, b^3 = b >.$$

Mentally think, a=nim heap of size 1, b=nim heap of size 2.

$T_2 = \{$	$1,$	$a,$	$b,$	$ab,$	$b^2,$	ab^2 $\}$
	0^{120}	1^{031}	2^{20}	3^{31}	0^{02}	1^{13}
	↓	↓	↓	↓	↓	↓
	even #'s	↓	$*2$	$*3$	$*0$	$*1$
		odd #'s			↓	↓
					(w/	at least 1 heap of size > 1)

6 elements (4 + 2 elements from T_1)

$\mathcal{P}$-positions are to a and b^2 (positions where 1st superscript is 0).

Φ	1	2	3	4	5
0+	a	1	b	b	1
5+	b^2	1			
10+					

⋆ Once you let anything wild in the door, you can't trust/use genera in the door!

4.3 Questions

- $Q_7(0.123) = T_2$. After heap size 7, then what??
- What's the analog of the mex function?
- Is our quotient still correct?

Example 12

$$\begin{matrix} 3 & 4 & 6 & \\ b & b & b^2 & = b^4 = b^3 b = b^2 \in \mathcal{P} \end{matrix}$$

$$\begin{matrix} 1 & 4 & 6 & \\ a & b & b^2 & = ab^3 = ab \in \mathcal{N} \end{matrix}$$

From 1 4 6, how do we find a move to a $\mathcal{P}$-position (i.e. a move to a or b^2)?

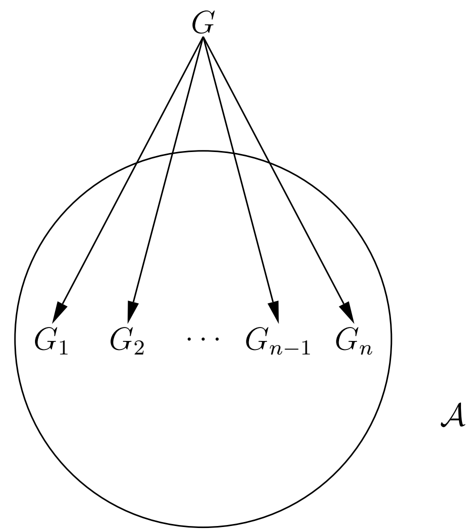
Move to

$$\begin{matrix} 1 & 1 & 6 & \\ a & a & b^2 & = a^2 b^2 = b^2 \in \mathcal{P} \end{matrix}$$

4.4 Transition Algebras

Let $\mathcal{A}$ be a set of games closed under moves, +. Suppose Q is its misere quotient.

$G \notin \mathcal{A}$.



Is $Q(\mathcal{A} \cup \{G\}) = Q(\mathcal{A})$? If so, can we determine $\Phi(G)$ from $\Phi(G_1), \ldots, \Phi(G_n)$, where $G_1, \ldots, G_n$ are options of G?

YES! The transition algebra $T(\mathcal{A})$ is the element we need.

Claim 1
There is a mapping (particular function) $F : Pow(Q) \to Q$ such that

(1) $F(S)$ *is defined* $\iff Q(\mathcal{A} \cup G) = Q(\mathcal{A})$

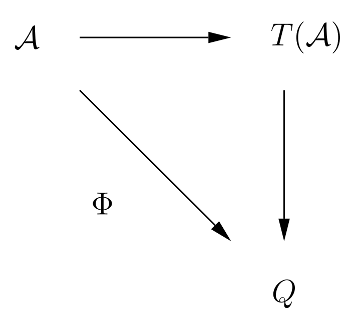
(2) *When its defined, necessarily* $\Phi(G) = F(\Phi'(G))$.

Definition 1 *The transition algebra, denoted $T(\mathcal{A})$, is defined as*

$$T(\mathcal{A}) = \{(\Phi(H), \Phi(H')) \mid H \in \mathcal{A}\}$$

where H' denotes options of the game H.

Example 13
Previously, we looked at 3 4 6 $= b^2$ which had a move to 1 4 6 $= b^2$, so (b^2, b^2) is an element of $T(\mathcal{A})$.



Note that we should have the complete normal play solution prior to looking at the misere version.

Normal Kernel Theorem $\Longrightarrow$ misere version is always at least as difficult as the impartial normal play version.

TUESDAY, 22 AUGUST 2006 - 9:30AM

5 Paul Ottaway: Misere Outcome Classes

Nothing works out nicely with misere outcome classes.

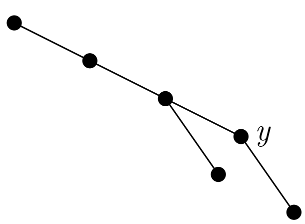
5.1 Equivalence Classes

$G \approx H$, G and H in same outcome class.

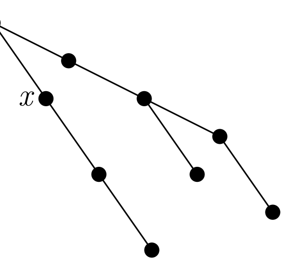
Do there exist G, H such that $G + K \approx H + K$, $\forall K$? (G and H different in some meaningful way)

Example 14

$G =$



$H =$



Pick some game K. Assume Right can win $G + K$ playing 1st (2nd). Then Right can win $H + K$ playing 1st (2nd).

Assume Left can win $G + K$ playing 1st. If Right can win $H + K$ playing 2nd, he must eventually play x (or he would have had a winning move in G). If moving to x were a winning strategy, then he could have won $G + K$ by ignoring the branch at y.

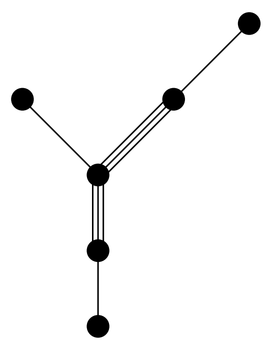
$\therefore G + K \approx H + K \quad \forall K.$

[More to come from Paul - already put into LaTeX for upcoming paper.]

TUESDAY, 22 AUGUST 2006 - 11:20AM

6 Richard Nowakowski: Sequential Joins

6.1 Childish Hackenbush Model



$G \in \mathcal{R}$

Definition 2
The sequential join of G and H is defined by

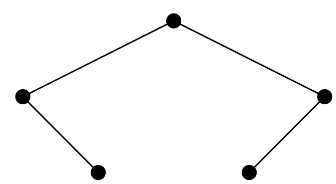
$$G \triangleright H = \begin{cases} \{G^L \triangleright H | G^R \triangleright H\} & \text{if } G \neq \{\,|\,\} \\ H & \text{otherwise} \end{cases}$$

Can't play in H before exhausting all possibilities in G.

Add from table	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{R}$
$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{R}$
$\mathcal{P}$	*Any*	*Any*	*Any*	*Any*
$\mathcal{L}$	$\mathcal{L}/\mathcal{N}$	$\mathcal{L}/\mathcal{P}$	$\mathcal{L}$	*Any*
$\mathcal{R}$	$\mathcal{R}/\mathcal{N}$	$\mathcal{R}/\mathcal{P}$	*Any*	$\mathcal{R}$

Identity? $e \triangleright G \approx G \triangleright e \approx G, \;\forall G$

Yes!



Also,

$$\overline{\mathcal{L}} = \Big\{G \,|\, \exists K, H \in \overline{\mathcal{L}}, \quad G = \{\,|\,\{\,|\,\}\} \text{ or } \{\ |K\} \text{ or } \{H|K\} \text{ or } \{H|\{\,|\,\}\}\Big\}$$

$$\overline{\mathcal{R}} = \overline{\mathcal{L}}$$

$$\overline{\mathcal{N}} = \Big\{G| \text{ if } G^L \text{ exists, } G^{LR} \in \overline{\mathcal{N}} \text{ and } G^{LL} \in \overline{\mathcal{L}} \\ \text{if } G^R \text{ exists, } G^{RL} \in \overline{\mathcal{N}} \text{ and } G^{RR} \in \overline{\mathcal{R}}\Big\}$$

If $X \in \overline{\mathcal{N}}$ and both players have a move, then

$$X \triangleright G \approx G \triangleright X \approx G, \;\forall G$$

Example 15

1-D Clobber with Left-end Sequential Join:

When the game splits, always play in the left most component. ($\longleftarrow$ Left / Right $\longrightarrow$)

x	o	x	o	x	o	x	o	x

6.2 Questions

- What is the set $\{\, e \mid e \triangleright G \;\approx\; G \triangleright e \;\approx\; G,\ \forall G\}$ (i.e. the set that forms the identity)?
- Is the set of identities for normal play equal to the set for misere play?

TUESDAY, 22 AUGUST 2006 - 11:40AM

7 Thane Plambeck: Birthdays for Partisan Misere Games

7.1 Problem

Understand birthday 2 for partisan misere games (disjoint sum)

7.2 Birthdays 0 and 1

Birthday	Games	
0	$\{\mid\}$	
1	$\{0\mid\}$	$= 1$
	$\{\mid 0\}$	$= -1$
	$\{0\mid 0\}$	$= *$

No Stars $1 \setminus -1$	0	1	2	3	4	5
	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
	$\mathcal{R}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{L}$
One Star						
	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{L}$

7.3 Birthday 2

$\{G|H\}$ where $G, H \in \{0, 1, -1, *\}$.

So there are most $16 \times 16 = 256$ games with birthday 2.

$G \rho H$ at birthday 2

$$o(G + X) = o(H + X), \qquad x \in \mathcal{A}$$

Can we understand all the sums? If not, which can we do?

TUESDAY, 22 AUGUST 2006 - 1:00PM

8 Impartial Misere Clobber - notes from board

game	genus (options)	
XO	1^{031}	
XXO	2^{20}	
XOX	1^{031}	
$XXXO$	1^{031}	
$XXOX$	2^{20}	
$XXOO$	0^{120}	
$XOXO$	3^{31}	
$XOOX$	0^{120}	
$XXXXO$	2^{20}	
$XXXOX$	1^{031}	
$XXXOO$	0^{120}	
$XXOXX$	2^{20}	
$XXOXO$	1^{031}	$(\{0^{120}, 2^{20}, 1^{031} + 1^{031}\})$
$XXOOX$	3^{31}	
$XOXXO$	0^{120}	
$XOXOX$	0^{120}	
$XOOOX$	1^{031}	
$XXXXOO$	0^{120}	
$XXXOXO$	4^{46}	
$XXXOOX$	0^{120}	
$XXXOOO$	0^{120}	
$XXOXXO$	0^{120}	
$XXOXOX$	3^{31}	
$XXOXOO$	1^{031}	
$XXOOXX$	0^{120}	
$XXOOXO$	1^{031}	
$XOXXXO$	3^{31}	
$XOXXOX$	0^{120}	
$XOXXOO$	1^{031}	
$XOXOXO$	0^{120}	
$XOOOXX$	1^{031}	
$XOOOOX$	0^{120}	
$XXXOOXX$	3^{31}	
$XXXOXXO$	2^{1420}	$(\{0^{02}, 1^{031}, 4^{46}\})$
$XXOXOXO$	3^{1431}	$(\{0^{02}, 1^{031}, 2^{20}\})$
$XXXOOXX$	3^{31}	
$XXXOOOX$	1^{20}	$(\{0^{02}, 0^{120}\})$
$XXOOXOX$	1^{431}	$(\{0^{02}, 0^{120}, 2^{20}, 3^{31}\})$
$XXXOXXOO$	1^{20}	$(\{0^{02}, 0^{120}\})$
$XXOXOOO$	1^{20}	$(\{0^{02}, 0^{120}\})$

WEDNESDAY, 23 AUGUST 2006

9 Partizan Misere Games - Notes from Board

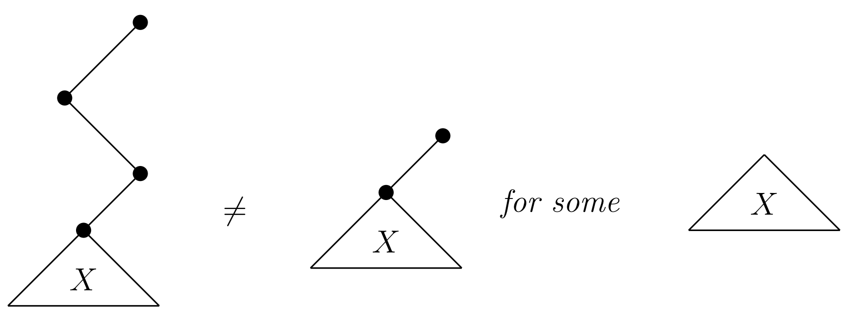
9.1 Domination

Definition 3
$G \geq H$
$\iff \forall K$, *Left wins* $H + K \implies$ *Left wins* $G + K$.
$\iff \forall K$, *Left wins* $H^L + K$ *going 2nd* $\implies$ *Left wins* $G + K$ *going 1st.*

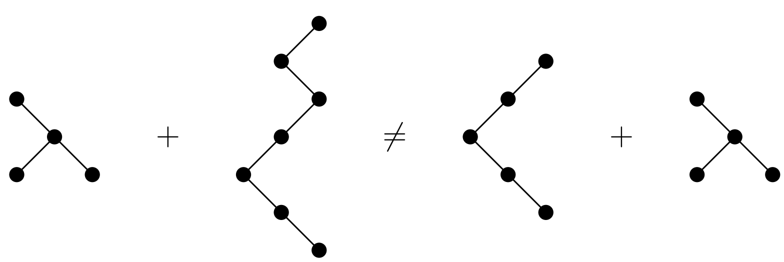
Defines a partial order on all games (normal or misere play). Reference?

Definition 4
For two games in misere play, $G \geq H$
if $(\emptyset \neq G^R \subseteq H^R$ *or* $G^R = H^R = \emptyset)$ *and* $(\emptyset \neq H^L \subseteq G^L$ *or* $G^L = H^L = \emptyset)$

Theorem 5



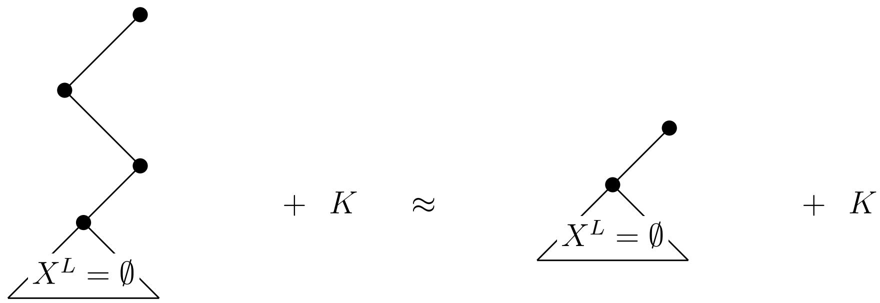
Proof:



■

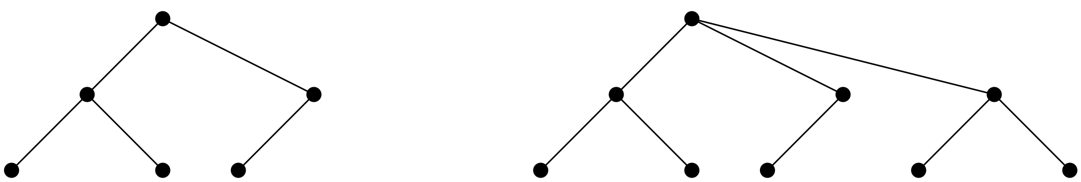
Conjecture 6

$\forall K$,

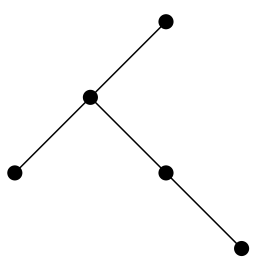


Claim 2 *The games* $\{*|\,1\}$ *and* $\{*|1,*\}$ *are distinguishable.*

The above games,



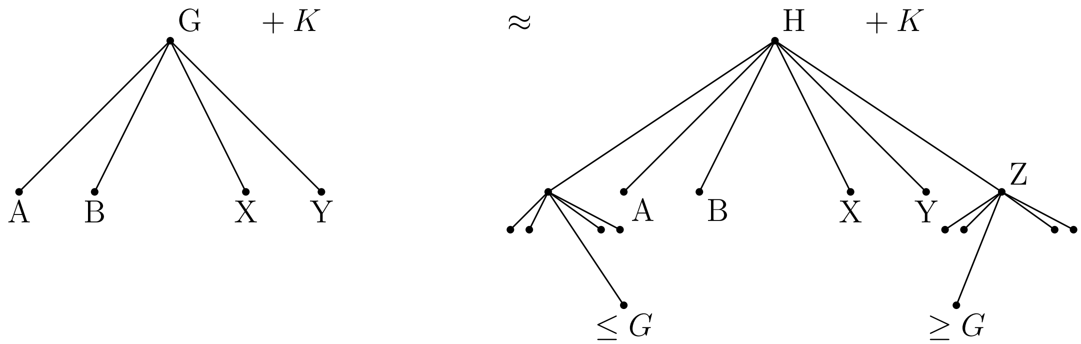
are distinguished by the game



9.2 Reversibility

General Idea:

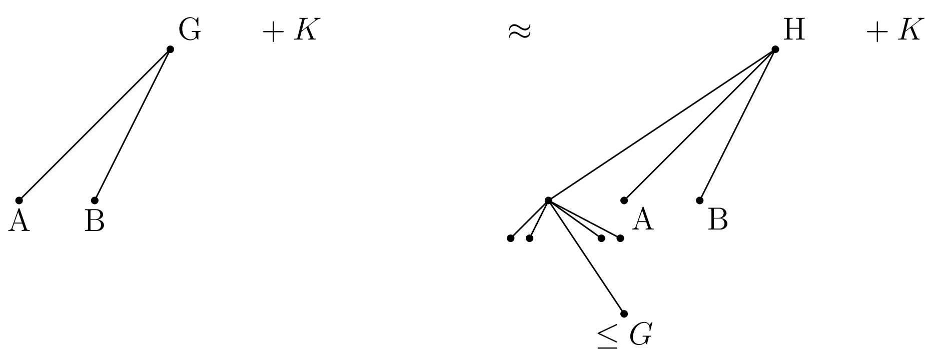
$\forall K$,



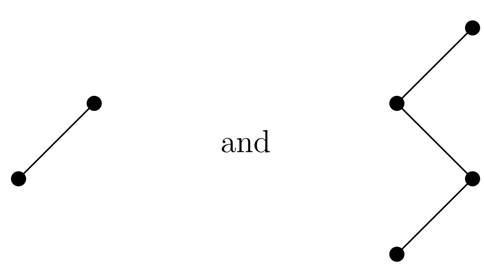
when $G^L, G^R \neq \emptyset$. Idea is that from H, all options in G are represented and all other moves are reversible to G.

Can only include reversible moves for Left (Right) if other moves exist for Left (Right).

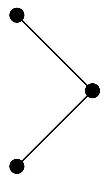
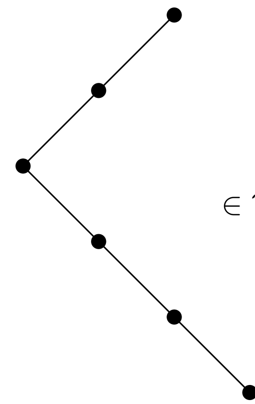
i.e.



Indistinguishable:



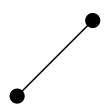
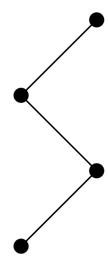
Distinguishable:

 $\neq$  as they are distinguished by  $\in \mathcal{R}$

Note: This is a counterexample for our general idea (stated above) when $G^L, G^R = \emptyset$.

Theorem 7

Let

A =  *and* B = 

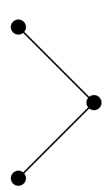
Then, $\forall K, A + K \approx B + K$.

Proof: Let K be minimal. If Left has a winning strategy playing first in $A + K$, then in order to win, he will eventually have to play to

 $+ \; K'$.

Since Left wins in this game, Right has some move in K'. Because it is Right's turn, it is still Left's win playing second.

In B, Left eventually must move to

 $+ \; K'$.

Similarly, Left must have a winning move playing second here. But then Right always has an option in both components on her turn (since we knew that she must have a move in K').

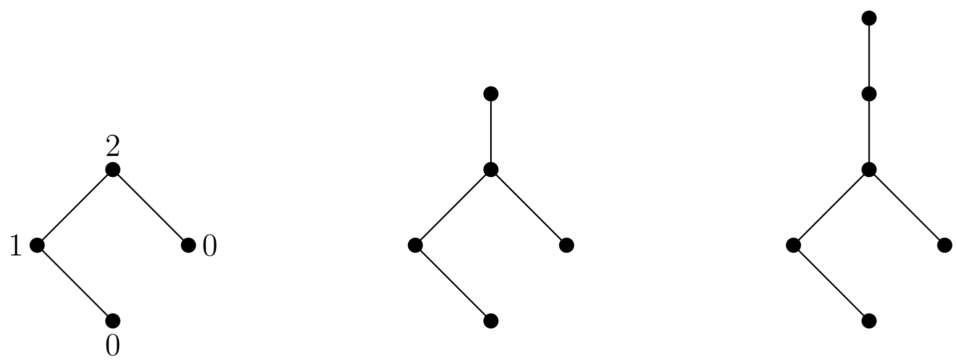
■

If (Left wins moving 1st in $H + K \implies$ Left wins moving 1st in $H + K$)
and (Left wins moving 2nd in $H + K \implies$ Left wins moving 2nd in $H + K$)
then $G \geq H$.

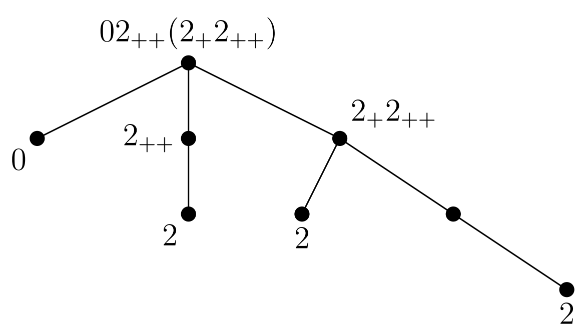
Wednesday, 23 August 2006

10 Transition Algebras - Notes from Work

Notation:



2 2_+ 2_{++}



10.1 0.123 Transition Algebra, $T(2_{++})$

To heap size 7:

- $T_2 =< a, b|\, a^2 = 1, b^3 = b >$
- $\mathcal{P} = \{a, b^2\}$

To heap size 8:

- $Q_8 =< a, b, c|\, a^2 = 1, b^4 = b^2, b^2c = b^3, c^2 = 1 >$
- $\mathcal{P} = \{a, b^2, ac\}$

To heap size 9:

- $Q =< a, b, c, d|\, a^2 = 1, b^4 = b^2, b^2c = b^3, c^2 = 1, b^2d = d, cd = bd, d^3 = ad^2 >$
- $\mathcal{P} = \{a, b^2, ac, bd, d^2\}$

Possible moves:

$$\begin{array}{rll}
h_1 \longrightarrow h_0 & & (a, 1) \\
h_3 \longrightarrow h_1 & & (b, a) \\
\longrightarrow h_0 & & (b, 1) \\
h_4 \longrightarrow h_2 & & (b, 1) \\
\longrightarrow h_1 & & (b, a) \\
h_5 \longrightarrow h_3 & & (a, b) \\
\longrightarrow h_2 & & (a, 1) \\
h_6 \longrightarrow h_4 & & (b^2, b) \\
\longrightarrow h_3 & & (b^2, b) \\
h_7 \longrightarrow h_5 & & (1, a) \\
\longrightarrow h_4 & & (1, b)
\end{array}$$

$$\begin{array}{ll}
A & = (1, \{a, b\}) \\
B & = (a, \{1, b\}) \\
C & = (b, \{1, a\}) \\
D & = (b^2, \{b\})
\end{array}$$

10.2 $(\mu, E) \times (\beta, F) = (\mu\beta, \beta E \cup \mu F)$

$$
\begin{aligned}
A &= (1, \{a, b\}) \\
B &= (a, \{1, b\}) \\
C &= (b, \{1, a\}) \\
D &= (b^2, \{b\}) \\
AA &= (1, \{a, b\}) \\
AB &= (a, \{1, b, ab\}) \\
AC &= (b, \{1, a, ab, b^2\}) \\
AD &= (b^2, \{b, ab^2\}) \\
BB &= (1, \{a, ab\}) \\
BC &= (ab, \{1, a, b, b^2\}) \\
BD &= (ab^2, \{b, ab, b^2\}) \\
CC &= (b^2, \{b, ab\}) \\
CD &= (b, \{b^2, ab^2\}) \\
DD &= (b^2, \{b\}) \\
ABC &= (ab, \{1, a, b, b^2, ab^2\})
\end{aligned}
$$

Q: Is $T(*2) \cong T(2_+)$?

$Q(*2) \cong Q(2_+)$

Normal play:

$G = *3 = \{*0, *1, *2, *4, *5, *6, *7\} = \{*0, *1, *2\}$.

Generalized Mex Rule:

Let $T = T(\mathcal{A})$ and $G \neq 0$ be options $\subset \mathcal{A}$.

(a) $Q(\mathcal{A} \cup G) = Q(\mathcal{A})$

(b) $\Phi''(x) \subset M_x$
For each $(z, E) \in T$ and each $n \geq 0$ such that $x^{n+1} \notin \mathcal{P}$ we have either

- (i) $x^{n+1}w \in \mathcal{P}$ for some w or else
- (ii) $x^n yz \in \mathcal{P}$ for some $y \in \Phi''(G)$.

The next heap is h_8.

The moves are

$$h_8 \to h_6$$
$$\to h_5$$

Option set (E) is to $\{a, b^2\}$.

Somehow, theorem 7.5 fails for every choice of a putative $\Phi(h_8) = x \in Q$.

Based on normal play nim values, we might guess that $\Phi(h_8) = b$. We're going to see that this fails.

Thm 7.5: For each (z, E) and each $n \geq 0$ such that $x^{n+1}z \in \mathcal{P}$ ...

(For us, $x = b$) Look at all choices of $z \in Q$ (six of them):

z	b	b^2
1	b	b^2
a	ab	ab^2
b	b^2	b
ab	ab^2	ab
b^2	b	b^2
ab^2	ab	ab^2

THURSDAY, 24 AUGUST 2006

11 Partizan Misere Games (Take 2) - Notes from Board

11.1 Domination (Take 2)

Definition 8

$G \geq H$ if

$(G^L = H^L = \emptyset$ *or* $G^L \supseteq H^L \neq \emptyset)$

and

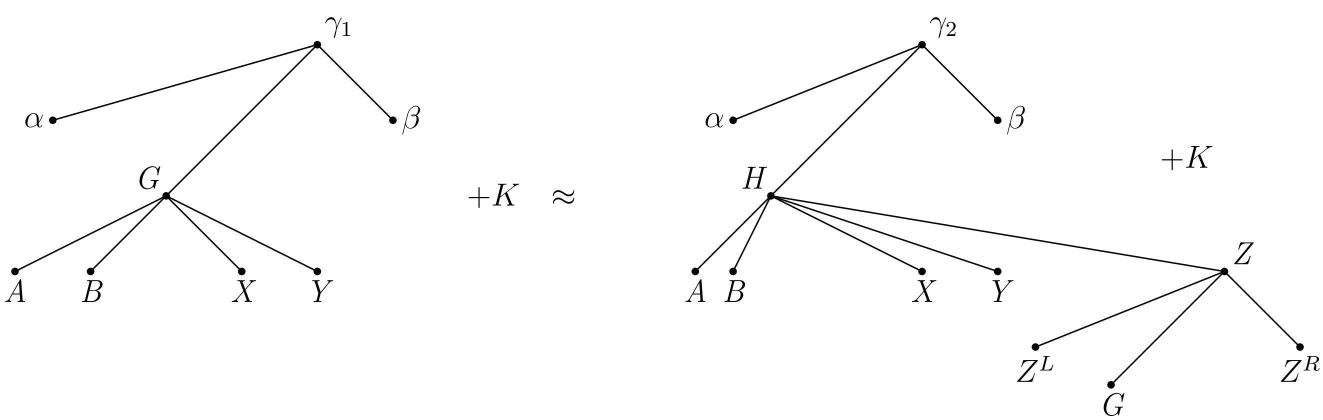
$(H^R = G^R = \emptyset$ *or* $H^R \supseteq G^R \neq \emptyset)$.

11.2 Reversibility (Take 2)

NOTE: We think that this is trumped!

Claim 3

$\forall K,$



PROVISO: If $X = Y = \emptyset$, then $Z^L = \emptyset$. ***(NOTE: This is NOT enough!)***

Must show:

(1) Left wins $\gamma_1 + K$ going first $\implies$ Left wins $\gamma_2 + K$ going first and
(2) Right wins $\gamma_1 + K$ going first $\implies$ Right wins $\gamma_2 + K$ going first.

or, equivalently,

(1) $\gamma_1 + K \in \mathcal{L} \cup \mathcal{N} \implies \gamma_2 + K \in \mathcal{L} \cup \mathcal{N}$ and
(2) $\gamma_1 + K \in \mathcal{R} \cup \mathcal{N} \implies \gamma_2 + K \in \mathcal{R} \cup \mathcal{N}$.

Proof:

(1) Assume that Left can win $\gamma_1 + K$ going first. Left plays the same winning strategy in $\gamma_2 + K$ as he would in $\gamma_1 + K$, moving to H instead of G, if needed. If Right never plays to Z, she loses since she loses in $\gamma_1 + K$. Otherwise, we have K' such that $G + K' \in \mathcal{L} \cup \mathcal{P}$.

Then in $\gamma_2 + K$, if Right moves from $H + K'$ to $Z + K'$ then Left responds to $G + K' \in \mathcal{L} \cup \mathcal{P}$ and wins.

$\therefore$ Left wins $\gamma_2 + K$ playing first.

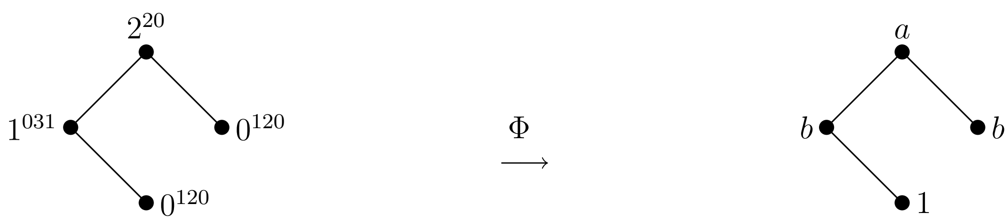
(2) Right ignores the move to Z.

∎

FRIDAY, 25 AUGUST 2006

12 Transition Algebras: a look at $T(2)$ - Notes

Notation:



$a^2 = 1$
$b^3 = b$

$$
\begin{aligned}
1 &= 0^{120} \\
a &= 1^{031} \\
ab &= 3^{31} \\
ab^2 &= 1^{13} \\
b &= 2^{20} \\
ab^2 &= 0^{02}
\end{aligned}
$$

In normal play, $G^+(G) = 0$ iff it is a $\mathcal{P}$ position. In misere, $G^-(G) = 0$ iff $\mathcal{P}$.

$\mathcal{P} = \{a, b^2\}$.

Possible moves:

$(a, 1)$
$(b, 1)$
(b, a)

$A = (a, \{1\})$
$B = (b, \{1, a\})$

12.1 $(\mu, E) \times (\beta, F) = (\mu\beta, \beta E \cup \mu F)$

e.g.

$$A \times A = (a, \{1\}) \times (a, \{1\}) = (a^2, \{a\}) = (1, \{a\})$$

$$A \times B = (a, \{1\}) \times (b, \{1, a\}) = (ab, \{a, a^2, b\}) = (ab, \{1, a, b\})$$

12.2 Pair translates or Elements of the transition algebra

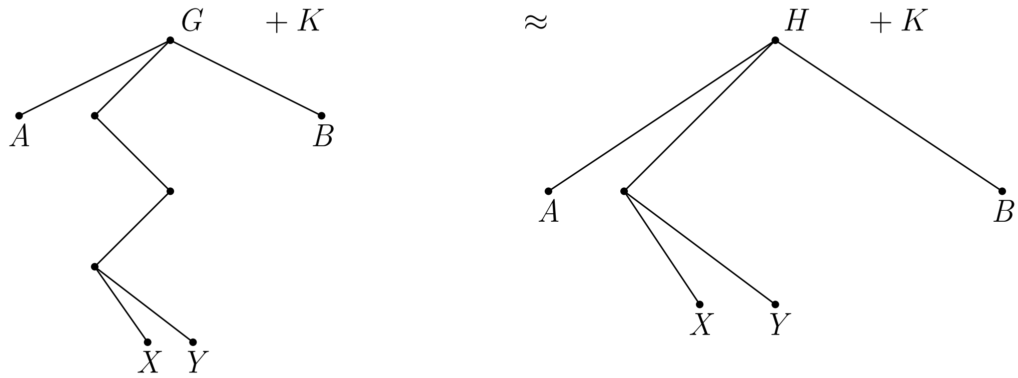
$$\begin{array}{cl} A & = (a, \{1\}) \\ B & = (b, \{1, a\}) \\ AA & = (1, \{a\}) \\ AB & = (ab, \{1, a, b\}) \\ \vdots & \end{array}$$

Friday, 25 August 2006

13 Partizan Misere Games (Take 3) - Notes from Board

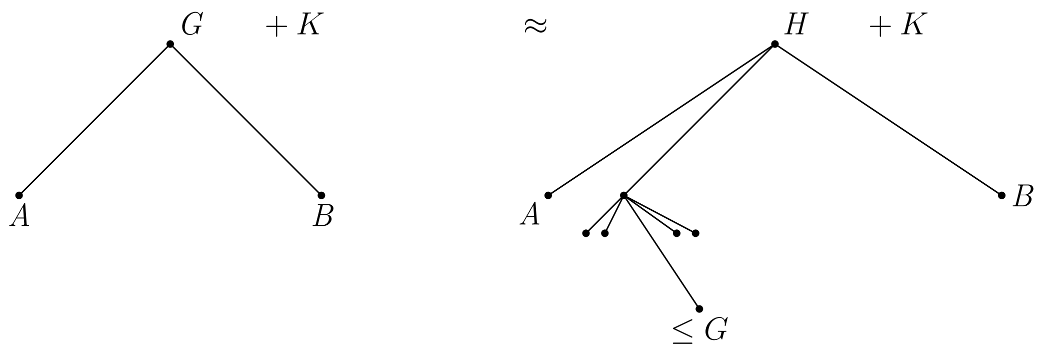
Theorem 9

$\forall K,$


G
+ K
A
B
X
Y
≈
H
+ K
A
B
X
Y

Theorem 10

$\forall K$,



where $A \neq \emptyset$.

Conjecture 11

Suppose $G + K \approx H + K \quad \forall K$ where G, H minimal with respect to domination and reversibility. Then G is identical to H (i.e. $G = H$).