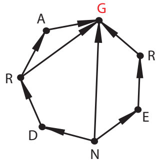
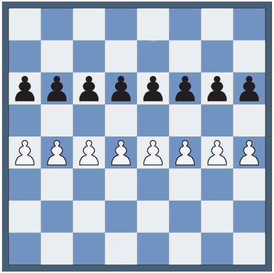
Advances in Losing

Thane Plambeck

Gathering for Gardner 7
Atlanta, March 2006



Dawson's Chess (1934)—a misere game



No. 30. Problemist Fairy Supplement, 1934. Given two equal lines of opposing Pawns, White on 3rd rank, Black on 5th, n adjacent files, White to play, at losing game, what is the result?

This recondite analysis may be commended to any mathematician. For small values of n, up to at least 50, first player loses if n equals 1, 2, 6, 7, or 11, modulus 14. In the case of remainder 4, mod. 14, the first player wins whatever move he plays first, e.g. for cases of 4, 18, and 32 files. For two separated groups of files, m and n, again for smaller values, the first player loses if to mod. 14, m and n are both 3, 5, 10, or 12; or if one is 1, 2, 6, 7, 11, and the other is 4, 8, 9, 13, or 14. But it is reasonably clear to me that the general case involves an infinity of expanding moduli, and the results to mod. 14 cannot be taken very far, probably not into the hundreds.

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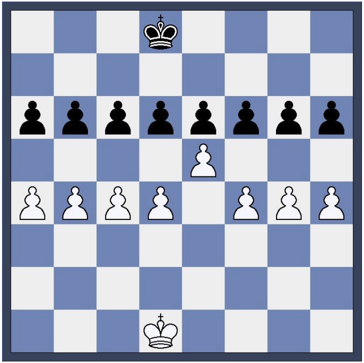
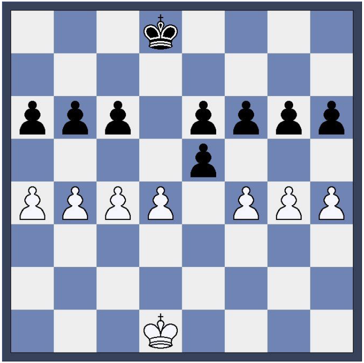
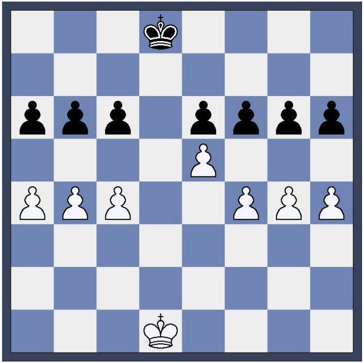
Play of Dawson's Chess



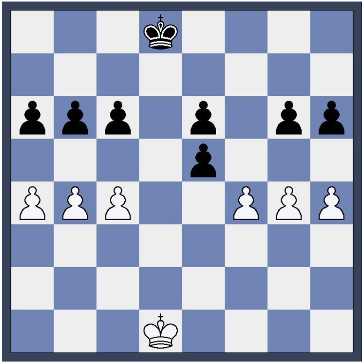
Play of Dawson's Chess



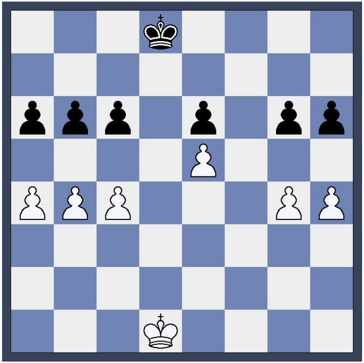
Play of Dawson's Chess



Play of Dawson's Chess



Symmetry: Dawson's Chess is an *impartial game*



Summary

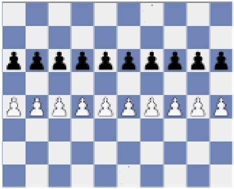
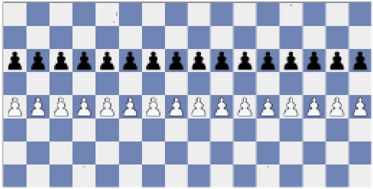
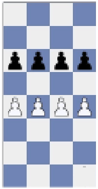
Dawson's chess is an *impartial game*.
Each position is either

- An **N-position**, ie, forced win for the *Next player* to move; or
- A **P-position**, ie a forced win for the *Previous player* to move.

It's played using the *misere play* convention—last player loses.

Sums in Dawson's chess

Is there a magic rule that tells us who should win?

$D_{10} + D_{16} + D_4$

Guy & Smith (1956)

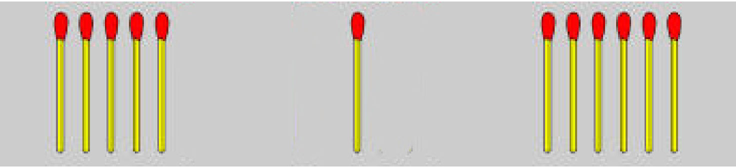
There is a magic rule for *normal play* of Dawson's chess.
Its *nim sequence* has period 34

	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3
0+	0	1	1	2	0	3	1	1	0	3	3	2	2	4	0	5	2	2	3	3	0	1	1	3	0	2	1	1	0	4	5	2	7	4
34+	0	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	2	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
68+	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
102+	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
136+	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
170+	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4

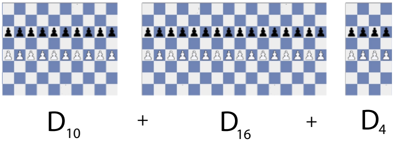
How should we use this information?

Nim

$$
\begin{aligned}
5 &= (101)_2 \\
1 &= (001)_2 \\
6 &= (110)_2 \\
&\quad - - - \\
5 \oplus 1 \oplus 6 &= (010)_2 \quad [\oplus = \text{ bitwise XOR (“add without carrying”)}]
\end{aligned}
$$



Normal play Dawson's chess is nim in disguise



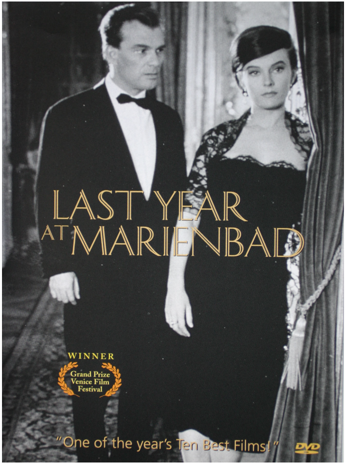
Use *nim addition* $\oplus$ to determine who wins $D_4 + D_{10} + D_{16}$:

$$0 \oplus 3 \oplus 2 = 1$$

It's a first-player win.
That shows how to play *normal play* Dawson's chess, using knowledge of how to play normal play Nim.

But that wasn't the problem—we're interested in *misere play*, instead.

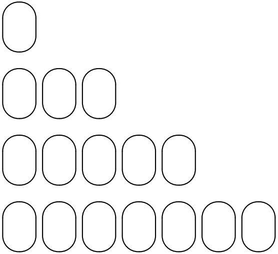
Misere Nim in the new wave French cinema (1961)



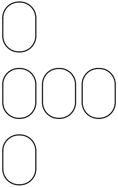
I am an avid fan of foreign, avant-garde, bizarre, challenging and/or enigmatic films, but this one is just plain agonizing to watch. The photography and the characters are beautiful, but I had to view this film in two sessions, both of them tormentingly slow. At first I thought it was some kind of variation on Sartre's "No Exit," but if it was, I was the one in the waiting room in Hell! This movie is pointless, vapid and pathetically pretentious. I hope God adds ninety-four minutes onto my life as a reward for sitting through Last Year at Marienbad!

---Jack M. Walter
Amazon customer review

Two Marienbad positions



Start position
$1 \oplus 3 \oplus 5 \oplus 7 = 0$



Misere exception to normal play strategy

The Big Picture

Impartial Games

Normal play

Misere play

Sprague-Grundy Theory
(nim heaps, nim addition)
1930s

Conway's Genus Theory
(tame games)
1970s

Wild games
(quotient monoids)
2005

How to generalize Sprague-Grundy to misere play

Γ = impartial game (fixed rules); $\mathcal{A}$ = set of all positions in Γ

$\Downarrow$

ρ = indistinguishability congruence on $\mathcal{A}$

$\Downarrow$

$\mathcal{Q}(\Gamma) = \mathcal{A}/\rho$ = quotient of all positions by ρ

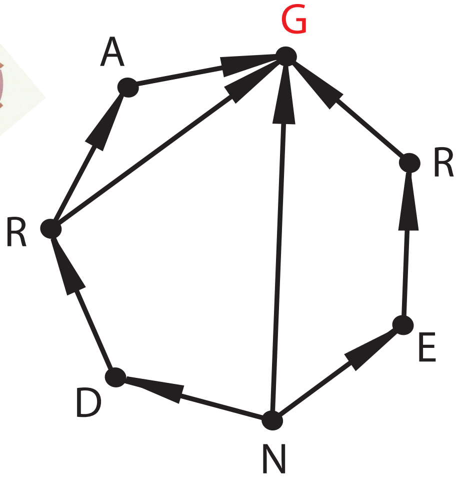
Normal play	Misere play
Sprague-Grundy Theory	Misere Quotient
$\mathcal{Q}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$	$\mathcal{Q}(\Gamma) = \langle$abelian monoid$\rangle$

Aaron Siegel's *MisereSolver*

Java language program that **calculates & exhibits** misere indistinguishability quotients Q for many wild misere games.

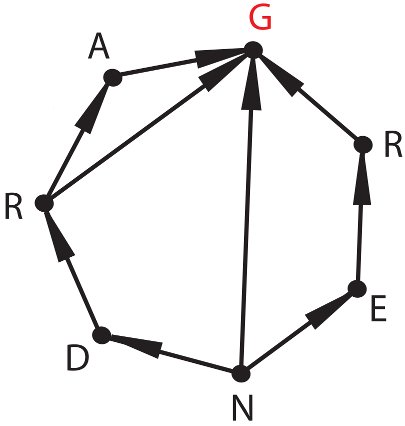
!!!

G4G7 gift exchange game: G-A-R-D-N-E-R

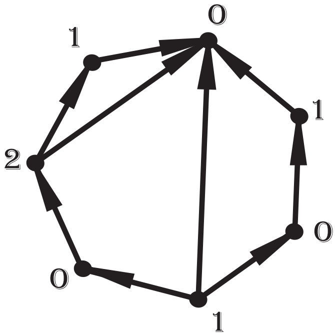


Normal or misere play?

Normal play—player to slide the last coin to the goal **wins**.
Misere play—player to slide the last coin to the goal **loses**.

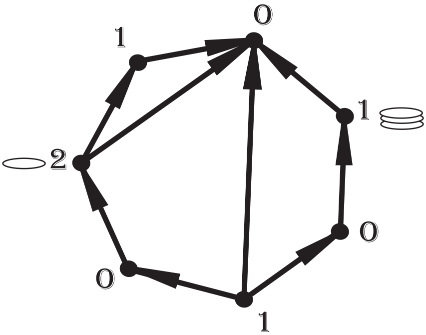


Normal play, part I: compute nim heap equivalents



1) Label the top node 0.
2) Use the mex rule to label the rest.

Normal play, part II: nim addition



$(01)_2$
$(01)_2$
$(01)_2$
$(10)_2$

nim sum $= (11)_2$

Make a move so that the recomputed nim-sum becomes $(00)_2$

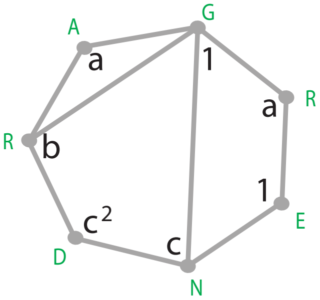
Misere play of GARDNER

Are there values we can similarly assign to solve misere play?
YES!—but:

- they aren't nim heaps;
- their addition isn't nim addition.

Instead the key is a 14-element monoid Q, the *misere indistinguishability quotient* of GARDNER.

GARDNER vertices are assigned monomials



The misere quotient Q of GARDNER

Introduce reduction rules given by the *abelian monoid presentation*

$$\mathcal{Q} = \langle a,\ b,\ c \mid a^2 = 1,\ b^3 = b,\ b^2c = c,\ c^3 = ac^2 \rangle.$$

The monoid $\mathcal{Q}$ = the GARDNER misere quotient.
A general monomial $a^i b^j c^k$ will always reduce to one of fourteen monomials

$$\mathcal{Q} = \{1,\ a,\ b,\ c,\ ab,\ ac,\ b^2,\ ab^2,\ bc,\ abc,\ abc^2,\ c^2,\ ac^2,\ bc^2\}$$

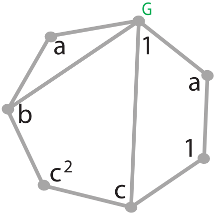
Four elements of $\mathcal{Q}$ correspond to P-position types:

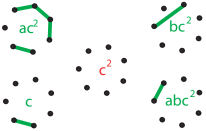
$$P = \{a, b^2, c^2, bc\}$$

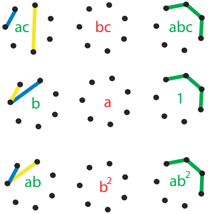
Reverse side of gift exchange item

$a^2 = 1$
$b^3 = b$
$b^2c = c$
$c^3 = ac^2$

green monomials = N-positions
red monomials = P-positions







- winning move
- winning for #b = 0 or 1
- winning for #b > 1

Normal play of *Guiles*

Rules of **Guiles**: You can

- remove a heap of 1 or 2 beans completely; or
- take two beans from a sufficiently large heap and partition what is left into two smaller, nonempty heaps.

	1	2	3	4	5	6	7	8	9	10
0+	1	1	0	1	1	2	2	1	2	2
10+	1	1	0	1	1	2	2	1	2	2
20+	1	1	0	1	1	2	2	1	2	2
30+	1	1	0	1	1	...				

Figure: Nim values for normal play Guiles

Misere play of *Guiles*

	1	2	3	4	5	6	7	8	9	10
0+	a	a	1	a	a	b	b	a	b	b
10+	a	a	1	c	c	b	b	d	b	e
20+	c	c	f	c	c	b	g	d	h	i
30+	ab^2	abg	f	abg	abe	b^3	h	d	h	h
40+	ab^2	abe	f^2	abg	abg	b^3	h	d	h	h
50+	ab^2	abg	f^2	abg	abg	b^3	b^3	d	b^3	b^3
60+	ab^2	abg	f^2	abg	abg	b^3	b^3	d	b^3	b^3
70+	ab^2	ab^2	f^2	ab^2	ab^2	b^3	b^3	d	b^3	b^3
80+	ab^2	ab^2	f^2	ab^2	ab^2	b^3	b^3	d	b^3	b^3
90+	ab^2	ab^2	f^2	ab^2	ab^2	b^3	b^3	d	b^3	b^3
100+										

Figure: Misere equivalences for Guiles.

Misere addition in *Guiles*

Order 42 monoid

$$\mathcal{Q} = \langle\, a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i \quad\mid\quad a^2 = 1,\ b^4 = b^2,\ bc = ab^3,\ c^2 = b^2,\ b^2d = d,\ cd = ad,\ d^3 = ad^2,\ b^2e = b^3,\ de = bd,\ be^2 = ace,\ ce^2 = abe,\ e^4 = e^2,\ bf = b^3,\ df = d,\ ef = ace,\ cf^2 = cf,\ f^3 = f^2,\ b^2g = b^3,\ cg = ab^3,\ dg = bd,\ eg = be,\ fg = b^3,\ g^2 = bg,\ bh = bg,\ ch = ab^3,\ dh = bd,\ eh = bg,\ fh = b^3,\ gh = bg,\ h^2 = b^2,\ bi = bg,\ ci = ab^3,\ di = bd,\ ei = be,\ fi = b^3,\ gi = bg,\ hi = b^2,\ i^2 = b^2 \,\rangle.$$

$$P = \{\, a,\ b^2,\ bd,\ d^2,\ ae,\ ae^2,\ ae^3,\ af,\ af^2,\ ag,\ ah,\ ai \,\}.$$

Misere sprouts: 3 spots per biosphere

How to add?

Misere sprouts addition, 3 spots per biosphere

Order 52 monoid

$$\langle\, a,\ b,\ c,\ d,\ e \quad | \quad a^2 = 1, b^5 = b^3, b^2c = c, c^3 = c^2, b^3d = b^4, \\ cd = bc, d^2 = 1, c^2e = c2, b^2e^2 = c^2, ce^2 = c^2, e^3 = c^2 \,\rangle.$$

$$P = \{a, b^2, b^4, c, c^2, ad, be, ab^3e, abce, bde, e^2\}$$

Dawson's chess—at most 33 vs 33 pawns per summand (order = 638)

$$
\begin{aligned}
Q_{33} = \langle\, & a, b, c, d, e, f, g, h, i, j, k, l, m, n \mid a^2 = 1, b^5 = b^3, b^2c = b^3, c^3 = c, b^2d^2 = b^2, \\
& bcd^2 = bc, d^3 = d, b^2e = b^4, ce = b^3, e^2 = b^4, b^2f = ab^2, cf = ac, bf^2 = b, df^2 = d, ef^2 = e, f^3 = f, \\
& bg = abc, cdg = ac^2d, d^2eg = eg, f^2g = g, g^2 = acg, bch = b^4h, f^2h = h, cgh = ac^2h, b^3h^2 = bh^2, \\
& ch^2 = bh^2, d^2h^2 = h^2, eh^2 = b^2h^2, bfh^2 = abh^2, gh^2 = abh^2, h^3 = b^2h^2, b^2i = ab^4, bci = ab^4, \\
& cdi = abc^2d, ei = ab^4, f^2i = i, gi = aci, chi = ab^3h, bh^2i = abh^2, i^2 = b^4, bj = abe, cj = ab^3, \\
& dj = ade, ej = ab^4, f^2j = j, h^2j = ab^2h^2, ij = b^4, j^2 = b^4, b^2k = b^4d, bck = b^4d, cdk = b^3, dek = b^4, \\
& dfk = adk, dgk = ab^3, egk = ab^3d, h^2k = b^2dh^2, bik = abek, dik = ab^4, hik = aehk, gjk = bek, \\
& bk^2 = b^3, dk^2 = b^4d, ek^2 = b^4, fk^2 = ac^2k^2, c^2gk^2 = gk^2, hk^2 = b^4h, c^2ik^2 = ik^2, jk^2 = ab^4, \\
& c^2k^3 = k^3, ik^3 = c^2ik, k^4 = c^2k^2, b^2l = ab^2d, cl = acd, f^2l = l, dgl = cd^2, egl = b^3d, bd^2hl = bhl, \\
& d^2ehl = ehl, h^2l = ab^2dh^2, bil = abd^2el, hil = aehl, jl = ael, bkl = abdk, dkl = ad^2k, ekm = bek \\
& ehkl = ab^4h, ikl = aekl, k^2l = ab^4d, d^2l^2 = l^2, el^2 = dk, fl^2 = al^2, gl^2 = acd^2, kl^2 = d^2k, \\
& l^3 = adl^2, bm = b^2, dm = bd, c^2gm = gm, egm = ab^4, fgm = agm, hm = bh, im = bi, jm = aem, \\
& c^2k^2m = k^2m, gk^2m = ik^2, lm = bd^2l, m^2 = b^2, b^2n = abd^2k, cn = ac^2d, bdn = il^2, den = ab^3, \\
& dfn = adn, gn = c^2d, bhn = ad^2hk, ehn = ab^3dh, h^2n = abdh^2, in = bc^2d, jn = aen, dkn = ab^3d, \\
& ekn = ab^3, bfkn = abkn, hkn = aghkl, k^3n = ab^3, dln = ad^2n, beln = aekl, hln = adhn, \\
& kln = b^3d, l^2n = d^2n, emn = ab^4d, fmn = adil^2, k^2mn = ab^4d, n^2 = c^2d^2 \,\rangle
\end{aligned}
$$

P-positions, Dawson's chess (to 33 pawns)

$$
\begin{aligned}
P = \{ & a, b^2, b^4, c, ac^2, ad, cd, ac^2d, ad^2, cd^2, \\
& ac^2d^2, e, d^2e, f, df, d^2f, aef, ad^2ef, \\
& af^2, ag, cg, ac^2g, adg, ad^2g, fg, dfg, d^2fg, \\
& abh, ab^3h, abd^2h, eh, bdeh, d^2eh, bfh, \\
& bd^2fh, befh, bd^2efh, aefgh, h^2, b^2h^2, afh^2, \\
& ahi, adhi, ad^2hi, abfhi, abd^2fhi, \\
& fh^2i, aj, fj, ahj, fghj, k, ack, c^2k, dk, afk, \\
& f^2k, gk, acgk, c^2gk, afgk, achk, dhk, abehk, ghk, \\
& afghk, cik, ak^2, ac^2k^2, ack^3, gk^3, l, dl, \\
& d^2l, adel, afl, adfl, ad^2fl, defl, gl, afgl, \\
& bdhl, bdehl, abdfhl, defhl, akl, fkl, efkl, fhkl, \\
& fghkl, al^2, adl^2, abhl^2, ckm, agkm, cgkm, ck^2m, \\
& an, adn, ad^2n, fn, af^2n, kn, afkn, \\
& f^2kn, ak^2n, ln, aeln, afln \}
\end{aligned}
$$

Monomial equivalents for *n* pawns in a row, Dawson's chess (to 33 pawns)

n	1	2	3	4	5	6	7	8
0+	1	a	a	b	1	ab	a	a
8+	1	c^2g	ab	c	b	d	1	ad
16+	ac^2g	b	ac	ab	e	f	ae	g
24+	h	c	i	j	c^2	k	l	m
32+	n							

References

http://arxiv.org/abs/math.CO/0501315
http://arxiv.org/abs/math.CO/0603027

Kayles, normal play

	1	2	3	4	5	6	7	8	9	10	11	12
0+	1	2	3	1	4	3	2	1	4	2	6	4
12+	1	2	7	1	4	3	2	1	4	6	7	4
24+	1	2	8	5	4	7	2	1	8	6	7	4
36+	1	2	3	1	4	7	2	1	8	2	7	4
48+	1	2	8	1	4	7	2	1	4	2	7	4
60+	1	2	8	1	4	7	2	1	8	6	7	4
72+	1	2	8	1	4	7	2	1	8	2	7	4
84+	1	2	8	1	4	7	2	1	8	2	7	4
96+	...											

Figure: The nim sequence of normal play Kayles.

Kayles, misere play

Order 40 misere quotient

	1	2	3	4	5	6	7	8	9	10	11	12
0+	x	z	xz	x	w	xz	z	xz^2	v	z	zw	t
12+	xz^2	z	zwx	xz^2	v^2t	xz	z	xvt	wz^2	zw	zwx	wz^2
24+	f	z	g	xwz^2	wz^2	zwx	z	xz^2	g	zw	zwx	wz^2
36+	xz^2	z	xz	xz^2	wz^2	zwx	z	xz^2	g	z	zwx	wz^2
48+	xz^2	z	g	xz^2	wz^2	zwx	z	xz^2	wz^2	z	zwx	wz^2
60+	xz^2	z	g	xz^2	wz^2	zwx	z	xz^2	g	zw	zwx	wz^2
72+	xz^2	z	g	xz^2	wz^2	zwx	z	xz^2	g	z	zwx	wz^2
84+	xz^2	z	g	xz^2	wz^2	zwx	z	xz^2	g	z	zwx	wz^2
96+	...											

Figure: The pretending function of misere Kayles.