The Misère Mex Mystery

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Reprise: Misère Quotients (1)

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$$X + Z \text{ and } Y + Z \text{ have the same outcome.}$$

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$$X + Z \text{ and } Y + Z \text{ have the same outcome.}$$

Define the quotient $\mathcal{Q} = \mathcal{Q}(\mathscr{A})$ by

$$\mathcal{Q}(\mathscr{A}) = \mathscr{A} / \equiv_{\mathscr{A}}$$

and let $\Phi : \mathscr{A} \to \mathcal{Q}$ be the quotient map. (So $\Phi(X)$ is the equivalence class of X modulo $\equiv_{\mathscr{A}}$.)

Reprise: Misère Quotients (2)

If $\Phi(X) = \Phi(Y)$, then X and Y have the same outcome.

Let $\mathcal{P} = \{\Phi(X) : X$ is a $\mathscr{P}$-position$\}$.

We call $\mathcal{P}$ the *$\mathscr{P}$-partition* of $\mathcal{Q}(\mathscr{A})$. The structure $(\mathcal{Q}, \mathcal{P})$ is the *misère quotient* of $\mathscr{A}$.

Reprise: Misère Quotients (3)

Let Γ be a heap game. Let $\mathscr{A}$ be the *heap algebra*—a free commutative monoid on the countable set of generators $\{H_0, H_1, H_2, \ldots\}$.

Given the misère quotient $(\mathcal{Q}, \mathcal{P})$ and the single-heap values $\Phi(H_k)$, we can read off a strategy for Γ.

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Namely, to find the outcome of $H_i + H_j + H_k$ (say), we compute $\Phi(H_i), \Phi(H_j), \Phi(H_k) \in \mathcal{Q}$, and check whether $\Phi(H_i)\Phi(H_j)\Phi(H_k) \in \mathcal{P}$.

Reprise: Misère Quotients (4)

These techniques have revealed strategies for many previously-unsolved *octal games*.

For example, 0.15 (Guiles—"Guy's Kayles").

Either:

- Completely remove a heap of one or two tokens; *or*
- Remove exactly two tokens from a heap of size ≥ 4, splitting the remainder into exactly two non-empty heaps.

Misère Guiles (1)

$$\mathcal{Q} \cong \langle a,b,c,d,e,f,g,h,i \mid a^2=1,\ b^4=b^2,\ bc=ab^3,\ c^2=b^2,\ b^2d=d,\ cd=ad,\ d^3=ad^2,\ b^2e=b^3,\ de=bd,\ be^2=ace,\ ce^2=abe,\ e^4=e^2,\ bf=b^3,\ df=d,\ ef=ace,\ cf^2=cf,\ f^3=f^2,\ b^2g=b^3,\ cg=ab^3,\ dg=bd,\ eg=be,\ fg=b^3,\ g^2=bg,\ bh=bg,\ ch=ab^3,\ dh=bd,\ eh=bg,\ fh=b^3,\ gh=bg,\ h^2=b^2,\ bi=bg,\ ci=ab^3,\ di=bd,\ ei=be,\ fi=b^3,\ gi=bg,\ hi=b^2,\ i^2=b^2\rangle$$

$$\mathcal{P}=\{a,b^2,bd,d^2,ae,ae^2,ae^3,af,af^2,ag,ah,ai\}$$

Misère Guiles (2)

	0	1	2	3	4	5	6	7	8	9
0	1	a	a	1	a	a	b	b	a	b
10	b	a	a	1	c	c	b	b	d	b
20	e	c	c	f	c	c	b	g	d	h
30	i	ab^2	abg	f	abg	abe	b^3	h	d	h
40	h	ab^2	abe	f^2	abg	abg	b^3	h	d	h
50	h	ab^2	abg	f^2	abg	abg	b^3	b^3	d	b^3
60	b^3	ab^2	abg	f^2	abg	abg	b^3	b^3	d	b^3
70	b^3	ab^2	ab^2	f^2	ab^2	ab^2	b^3	b^3	d	b^3
80	b^3	ab^2	ab^2	f^2	ab^2	ab^2	b^3	b^3	d	b^3
90	b^3	ab^2	ab^2	f^2	$\cdots$					

Normal Guiles

	0	1	2	3	4	5	6	7	8	9
0	0	1	1	0	1	1	2	2	1	2
10	2	1	1	0	1	1	2	2	1	2
20	2	1	1	0	1	1	2	2	1	2
30	2	1	1	0	1	1	2	2	1	2
40	2	1	1	0	1	1	2	2	1	2
50	2	1	1	0	1	1	2	2	1	2
60	2	1	1	0	1	1	2	2	1	2
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Suppose we are given a sum of nim-heaps

$$H_i + H_j + H_k$$

In normal play, this is a $\mathscr{P}$-position just if the Grundy values sum to 0:

$$i \oplus j \oplus k = 0$$

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The same is true in misère play, *unless* all the heaps have size 1.

Proof: Play normal Nim until your move would leave only heaps of size 1. Then play to leave an *odd* number of heaps of size 1.

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So the strategies for normal and misère Nim coincide so long as there is an n-heap for some $n > 1$.

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Goal: find an analogous statement for other misère games.

Misère Nim (3)

The quotient for misère Nim is

$$\mathcal{Q} = \langle a, b, c, d, \ldots \quad | \quad a^2 = 1, \\ b^3 = b, c^3 = c, d^3 = d, \ldots, \\ b^2 = c^2 = d^2 = \cdots \rangle$$

$\mathcal{P} = \{a, b^2\}$, $\Phi(*) = a$, $\Phi(*2) = b$, $\Phi(*4) = c$, $\Phi(*8) = d$, $\ldots$

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$\mathcal{P} = \{a, b^2\}$, $\Phi(*) = a$, $\Phi(*2) = b$, $\Phi(*4) = c$, $\Phi(*8) = d$, $\ldots$

Crucial fact! b^2 is an *idempotent*: $b^2 \cdot b^2 = b^2$, and we have

$$\mathcal{Q} \setminus \{1, a\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots,$$

with identity b^2. This looks just like the "normal quotient."

Some Semigroup Theory (1)

Let $\mathcal{Q}$ be a commutative monoid.

$x \in \mathcal{Q}$ is an *idempotent* iff $x^2 = x$.

If $x, y \in \mathcal{Q}$, then x *divides* y iff $xz = y$ for some $z \in \mathcal{Q}$.

x and y are *mutually divisible* iff x divides y and y divides x.

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x and y are *mutually divisible* iff x divides y and y divides x.

Let x be an idempotent. Let

$$\mathcal{E} = \{y \in \mathcal{Q} : x, y \text{ are mutually divisible}\}.$$

Then $\mathcal{E}$ is a group, with identity x. Indeed, if $y \in \mathcal{E}$ and $yz = x$, then z serves as an inverse for y.

Some Semigroup Theory (2)

Now suppose $\mathcal{Q}$ is *finite*.

Let $z_1, z_2, \ldots, z_n$ enumerate the idempotents of $\mathcal{Q}$.

Put $z = z_1 \cdot z_2 \cdot z_3 \cdot \cdots \cdot z_n$.

Now z is an idempotent, and $z \cdot x = z$ (z *absorbs* x) for any idempotent $x \in \mathcal{Q}$.

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Let $\mathcal{K}$ be the group

$$\mathcal{K} = \{y \in \mathcal{Q} : z, y \text{ are mutually divisible}\}.$$

$\mathcal{K}$ is called the *kernel* of $\mathcal{Q}$.

Some Semigroup Theory (3)

There is a natural surjective homomorphism from $\mathcal{Q}$ onto $\mathcal{K}$:

$$x \mapsto x \cdot z$$

To see that $x \cdot z \in \mathcal{K}$: let $y = x \cdot z$. We must show y, z are mutually divisible. Clearly z divides y. Now consider

$$y, y^2, y^3, y^4, \ldots$$

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$$y, y^2, y^3, y^4, \ldots$$

Since $\mathcal{Q}$ is finite, eventually we must have $y^n = y^{n+k}$. But then y^{ik} is an idempotent ($ik > n$).

By definition of z, y^{ik} divides z, so y divides z.

For surjectivity, note that z is the group identity of $\mathcal{K}$.

Misère Nim (4)

Let's see how this works for misère Nim to some finite heap size (say $*15$):

$$\mathcal{Q} = \langle a, b, c, d \;\;\mid\;\; \begin{aligned} & a^2 = 1, \\ & b^3 = b, c^3 = c, d^3 = d, \\ & b^2 = c^2 = d^2 \rangle \end{aligned}$$

The only idempotents are 1 and b^2, so $z = 1 \cdot b^2 = b^2$.

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The only idempotents are 1 and b^2, so $z = 1 \cdot b^2 = b^2$.

$\mathcal{K} = \mathcal{Q} \setminus \{1, a\}$. As we saw earlier, this gives

$$\mathcal{K} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Misère Nim (5)

Multiplication by $z = b^2$ induces a mapping

$$\mathcal{Q} \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

This mapping sends every $x \in \mathcal{Q}$ to its Grundy value!

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In other words, multiplication by b^2 sends each $x \in \mathcal{Q}$ to its "normal-play residue" in $\mathcal{K}$.

If x is already in $\mathcal{K}$, then $b^2 \cdot x = x$, and the normal and misère outcomes coincide.

Misère Nim (6)

Remember the strategy for misère Nim:

> Play normal Nim until your move would leave only heaps of size 1. Then play to leave an odd number of heaps of size 1.

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We can now rephrase this:

> Play normal Nim until your move would leave a position outside of $\mathcal{K}$. Then pay attention to the fine structure of the misère quotient.

We now have the right framework for a generalization.

Misère Guiles (3)

Let $(\mathcal{Q}, \mathcal{P})$ be the misère quotient of Guiles.

The idempotents of $\mathcal{Q}$ are 1, b^2, d^2, e^2, f^2. Multiplying them gives

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The kernel $\mathcal{K}$ is the mutual divisibility class of d^2:

$$\mathcal{K} = \{d^2, ad^2, bd^2, abd^2\}.$$

$\mathcal{K} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

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	0	1	2	3	4	5	6	7	8	9
0	1	a	a	1	a	a	b	b	a	b
10	b	a	a	1	c	c	b	b	d	b
20	e	c	c	f	c	c	b	g	d	h
30	i	ab^2	abg	f	abg	abe	b^3	h	d	h
40	h	ab^2	abe	f^2	abg	abg	b^3	h	d	h
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	0	1	2	3	4	5	6	7	8	9
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10	bd^2	ad^2	ad^2	d^2	ad^2	ad^2	bd^2	bd^2	ad^2	bd^2
20	bd^2	ad^2	ad^2	d^2	ad^2	ad^2	bd^2	bd^2	ad^2	bd^2
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Misère Guiles (5)

If we identify

$$0 \sim d^2,\ * \sim ad^2,\ *2 \sim bd^2,\ *3 \sim abd^2,$$

then we find $\mathcal{K} \cong \{0, *, *2, *3\}$! As with misère Nim, multiplication by $z = d^2$ maps the misère quotient onto its normal residue.

Now $d^2 \in \mathcal{P}$, and $ad^2, bd^2, abd^2 \notin \mathcal{P}$. We conclude with a strategy for misère Guiles . . .

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Play normal Guiles until your move would leave
a position outside of $\mathcal{K}$. Then pay attention to
the fine structure of the misère quotient.

Other Games

This works for many, many, many games. If we are playing Γ on some massive number of heaps, we can forget all about misère-play complexities so long as the position remains “rich enough” to stay inside of $\mathcal{K}$.

Only when the environment “thins out” do we need to start paying attention.

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Only when the environment "thins out" do we need to start paying attention.

A cautionary tale:

$$\mathcal{Q} = \mathcal{Q}((2_+31)(2_+2)2_+3)$$

Here the identity of $\mathcal{K}$ is *not* a misère $\mathscr{P}$-position. So the normal and misère strategies do *not* coincide. However, it *is* still true that $\mathcal{K} \cong$ the normal quotient, as a group.

The Normal Embedding Conjecture (1)

Conjecture: In any misère quotient, $\mathcal{K}$ is isomorphic to the normal quotient.

It's true under the following two assumptions:

- (faithfulness) If $\Phi(X) = \Phi(Y)$, then X and Y have the same Grundy value.
- (regularity) $\mathcal{K}$ contains just one $\mathscr{P}$-position.

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- (regularity) $\mathcal{K}$ contains just one $\mathscr{P}$-position.

So the Conjecture is equivalent to the question: does there exist a non-faithful or irregular quotient?

The Normal Embedding Conjecture (2)

There is overwhelming experimental evidence for it . . . but little reason to believe.

The history of misère games is rife with such conjectures that later prove to be false.

In Fact:

Old Conjecture: If $\mathcal{Q}$ is finite, then every element of $\mathcal{Q}$ has period 1 or 2. (The *period* of x is the least k for which $x^{n+k} = x^n$, for some n.)

Based on new evidence, we've been forced to revise this slightly.

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New Conjecture: For every $k \geq 1$, there exists a quotient $\mathcal{Q}$ and an element $x \in \mathcal{Q}$ of period k.

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Based on new evidence, we've been forced to revise this slightly.

New Conjecture: For every $k \geq 1$, there exists a quotient $\mathcal{Q}$ and an element $x \in \mathcal{Q}$ of period k.

$$\mathcal{Q}((2_+30)(2_+210)3_+21)$$

has an element $x \neq 1$ satisfying $x^3 = 1$.

The Misère Mex Mystery

We are given $\mathcal{Q}(\mathscr{A})$, together with some game G whose options are in $\mathscr{A}$.

We compute $\Phi(G')$ for each such option G'. How can we predict $\Phi(G)$?

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In normal play it's just the Grundy mex.

In misère play, we need more information.

Transition Algebras (1)

Let $x = \Phi(G)$ and consider

$$\mathcal{E} = \Phi''G = \{\Phi(G') : G' \text{ is an option of } G\}.$$

We consider the pair $(x, \mathcal{E})$. Define $\Psi(G) = (x, \mathcal{E})$.

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The *misère quotient* is given by

$$\mathcal{Q}(\mathscr{A}) = \{\Phi(G) : G \in \mathscr{A}\}.$$

The *transition algebra* is given by

$$T(\mathscr{A}) = \{(\Phi(G), \Phi''G) : G \in \mathscr{A}\} = \{\Psi(G) : G \in \mathscr{A}\}.$$

Transition Algebras (2)

There is a natural multiplicative structure on $T(\mathscr{A})$.

$$(x,\mathcal{E})\cdot(y,\mathcal{F})=(xy,x\mathcal{F}\cup y\mathcal{E}).$$

We are essentially rephasing the definition of $+$.

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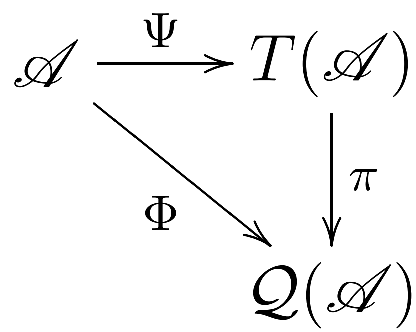
We are essentially rephasing the definition of $+$.

$$\Psi(G + H) = \Psi(G) + \Psi(H).$$

So $T(\mathscr{A})$ is another monoid, and the following diagram commutes:



Transition Algebras (3)

Given a game G with each option $G' \in \mathscr{A}$, let $\mathscr{B}$ be the closure of $\mathscr{A} \cup \{G\}$.

Given $T(\mathscr{A})$, we can instantly determine whether $\mathcal{Q}(\mathscr{B}) = \mathcal{Q}(\mathscr{A})$. If it does, we can instantly compute the value of $\Phi(G)$.

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The Normal Embedding Conjecture states that on the kernel $\mathcal{K} < \mathcal{Q}$, the normal and misère mex functions coincide. So if it's true, we can prove it (in theory) just by looking at transition algebras.

Transition Algebras (4)

Transition algebras also answer the following question. Given a monoid $\mathcal{Q}$ and a subset $\mathcal{P} \subset \mathcal{Q}$, is $(\mathcal{Q}, \mathcal{P})$ isomorphic to the misère quotient of some set $\mathscr{A}$?

But this is the subject of another talk.